

# Robust Inference for the Direct Average Treatment Effect with Treatment Assignment Interference

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## Abstract

Uncertainty quantification in causal inference settings with random network interference is a challenging open problem. We study the large sample distributional properties of the classical difference-in-means Hajek treatment effect estimator, and propose a robust inference procedure for the (conditional) direct average treatment effect, allowing for cross-unit interference in both the outcome and treatment equations. Leveraging ideas from statistical physics, we introduce a novel Ising model capturing interference in the treatment assignment, and then obtain three main results. First, we establish a Berry-Esseen distributional approximation pointwise in the degree of interference generated by the Ising model. Our distributional approximation recovers known results in the literature under no-interference in treatment assignment, and also highlights a fundamental fragility of inference procedures developed using such a pointwise approximation. Second, we establish a uniform distributional approximation for the Hajek estimator, and develop robust inference procedures that remain valid regardless of the unknown degree of interference in the Ising model. Third, we propose a novel resampling method for implementation of robust inference procedure. A key technical innovation underlying our work is a new *De-Finetti Machine* that facilitates conditional i.i.d. Gaussianization, a technique that may be of independent interest in other settings.

*Keywords:* causal inference under interference; Ising Model; Distribution Theory; Robust Inference

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# 1 Introduction

We study the large sample distributional properties of the classical Hajek average treatment effect estimator, and propose a robust inference procedure for the (conditional) direct average treatment effect, in the presence of cross-unit interference in both the outcome and treatment equations. This causal inference problem arises in a variety of contexts such as (online) social networks, medical trials, and socio-spatial studies, and has received renewed attention in recent years. Recent contributions include [1], [11], [10], [12], [18], [20], and references therein. See [8] for a modern textbook introduction to causal inference.

The key challenge in causal inference settings with interference is that units can affect each other in arbitrary ways, making statistical inference difficult without disciplining the degree of cross-unit interference: it is common to assume that units correspond to vertices in a network, typically represented as a graph, such that only when units are connected by an edge, they may influence each other. Early literature assumed that the underlying network was fixed, or otherwise known, but more recent advances have considered estimation and inference methods allowing the network to be a random (unobserved) graph (see Assumption 1 below). Furthermore, due to the challenges introduced by the presence of the latent random graph structure, it is common in the literature to restrict the degree of interference entering the outcome and treatment equations: prior work has focused on the special case where the potential outcomes exhibit restricted interference in the form of *anonymity* or *exchangability* (see Assumption 2 below), but the treatment assignment mechanism does not exhibit interference. We contribute to this emerging causal inference literature by allowing for the treatment assignment mechanism to also exhibit restricted cross-unit interference, while retaining the other semiparametric modelling assumptions imposed in previous work.

Leveraging ideas from statistical physics [7], we introduce a class of Ising equiprobable treatment assignment mechanisms described by

$$\mathbb{P}_\beta(\mathbf{T} = \mathbf{t}) \propto \exp\left(\frac{\beta}{n} \sum_{i \neq j} (2t_i - 1)(2t_j - 1)\right), \quad (1)$$

where  $\mathbf{T} = (T_1, \dots, T_n)^\top \in \{0, 1\}^n$  denote the vector of binary treatment assignments for  $n$  units,  $\mathbf{t} = (t_1, \dots, t_n)^\top$ , and the unknown parameter  $\beta \geq 0$  controls the degree of cross-unit interference in their treatment assignments (see Assumption 3 below). This model explicitly accounts for the stochastic nature of network formation in the treatment equation, and reduces to the classical independent equiprobable treatment assignment rule when  $\beta = 0$  (i.e., random assignment with equal probability). Thus, the Ising equiprobable treatment assignment model allow us to investigate how prior conclusions in the literature change as a function of the degree of cross-unit interference in treatment assignment as controlled by the unknown parameter  $\beta$ .

To streamline the presentation, and due to some technical issues, we focus on the moderate cross-unit interference regime  $\beta \in [0, 1]$ . See Section 8 for more discussion. Our first contribution concerns the large sample distributional properties of the classical difference-in-means Hajek estimator (see (3) below). Theorem 3.1 establishes a Berry-Esseen bound for the estimator, that is, a distributional approximation in Kolmogorov distance with explicit convergence rates. The closest antecedent is [12], who considered the same causal model

with network interference but under the assumption  $\beta = 0$  (random treatment assignment), and established a Gaussian distributional approximation for the Hajek estimator. Theorem 3.1 establishes a precise distributional approximation with explicit convergence rates and, more importantly, shows that: (i) for  $\beta \in [0, 1)$ , the limiting distribution continues to be Gaussian, but the asymptotic variance exhibits an additional term that captures the cross-unit interference in the treatment equation; (ii) for  $\beta \in [0, 1)$ , the new asymptotic variance coincides with the one obtained in [12] when  $\beta = 0$  but is increasing and unbounded as a function of  $\beta$ ; and (iii) for  $\beta = 1$ , the limiting distribution is non-Gaussian. These findings have an important implication for the robustness of inference procedures developed under the assumption of no-interference in the treatment assignment ( $\beta = 0$ ): the distribution approximation changes as a function of  $\beta \geq 0$ , exhibiting a discontinuity at  $\beta = 1$ , thereby invalidating inference procedures obtained from distributional approximations that only hold pointwise in  $\beta$ .

The lack of uniform validity demonstrated in Theorem 3.1 poses a major challenge for developing robust inference procedures in the presence of potential interference in the treatment assignment because  $\beta$  is unknown in practice. Moreover, [16] showed that no consistent estimator exists for  $\beta \in [0, 1)$ , making plug-in inference procedures infeasible, even pointwise in  $\beta \in [0, 1)$ . To address these challenges, Theorem 4.1 establishes a uniform in  $\beta \in [0, 1]$  distributional approximation for the Hajek estimator and, as a necessary by-product, also establishes a uniform distributional approximation in  $\beta \in [0, 1]$  for its Maximum Pseudo-Likelihood estimator (MPLE); see [17]. The resulting distributional approximations are indexed by a localization parameter offering a smooth transition between the discontinuous limit laws established in Theorem 3.1, as well as for those corresponding to the MPLE of  $\beta$ .

Building on Theorem 4.1, and employing a Bonferroni-correction procedure that works by creating hierarchical confidence intervals for different  $\beta$ -regimes, we present uniformly valid uncertainty quantification for the (conditional) direct average treatment effect  $\tau_n$  (see : we develop infeasible (Theorem 4.2) and feasible (Theorem 5.1) prediction intervals  $C_n(\alpha)$  satisfying

$$\liminf_{n \rightarrow \infty} \inf_{\beta \in [0, 1]} \mathbb{P}_\beta[\tau_n \in C_n(\alpha)] \geq 1 - \alpha,$$

for  $\alpha \in [0, 1]$ , where  $C_n(\alpha)$  is based on the Hajek estimator and a novel resampling procedure aimed to capturing sampling uncertainty coming from the underlying network. To the best of our knowledge, our proposed feasible inference procedure is new for  $\beta = 0$ . More importantly, our proposed inference procedure is the first to offer robust (uniform) validity across all values of  $\beta \in [0, 1]$ . Section 6 presents a simulation study demonstrating the performance of our proposed methods.

## 1.1 Summary of Methodological and Technical Contributions

From a methodological perspective, our paper contributes to the literature on causal inference under cross-unit interference. Classical contributions include [9], [19], [15], and references therein. The closest antecedent to our work is [12], who studied distribution theory for the same casual inference model with network interference considered in this paper except for

assuming random treatment assignment (i.e., without interference in the treatment assignment mechanism). Thus, our first methodological contribution is to propose a novel Ising equiprobable treatment assignment model to capture the possible interdependency between treatment assignments when units can interfere with each other. The model covers the equiprobable experimental design, as well as a class of dependent treatment assignments as indexed by  $\beta$  in (1). Furthermore, our second main methodological contribution is to present a novel, feasible robust inference procedure for the (conditional) direct average treatment effect, which is uniformly valid for all  $\beta \in [0, 1]$ . This procedure relies on a Bonferroni correction together with a uniform distributional approximation for the Hajek estimator, taking into account the different  $\beta$ -regimes, and also leverages a new resampling-based variance estimator developed herein. Our proposed inference procedure appears to be new even in the special case of  $\beta = 0$  (no-interference in treatment assignment).

From a technical perspective, our paper also offers a contribution to the applied probability literature, particularly in the context of statistical mechanics [7]. Allowing for interference in treatment assignment leads to major technical challenges for establishing distribution theory for the Hajek estimator, since the Ising equiprobable treatment model introduces new sources of dependence that need to be taken into account. For example, as shown in Theorem 3.1, the Hajek estimator exhibits different concentration rates around  $\tau_n$  depending on whether  $\beta = 1$  or not, in addition to having different limit laws. Our first technical contribution is to develop a new *De-Finetti Machine* that leverages the exchangeability structure in the treatment vector induced by Ising model, which we then use to establish a Berry-Esseen bound under the different  $\beta$ -regimes. This new technique is based on a carefully crafted conditioning argument that renders the elements of  $\mathbf{T}$  conditionally i.i.d., thereby reducing the problem to establishing a Berry-Esseen bound for conditionally i.i.d. random variables. Our new technical approach generalizes [5] and [6] by considering a multiplier Curie-Weiss magnetization statistic, without relying on variants of Stein’s method [5], and instead using a novel conditional i.i.d. Gaussianization approach. Our new technique may be of independent interest in other settings considering establishing a Berry-Esseen bound for sum of exchangeable random variables. To address the uniform inference problem, we further establish uniform in  $\beta \in [0, 1]$  distributional approximations: our results cover both the Hajek estimator and the MPLE for  $\beta$ . Thus, a second technical contribution of our work is to the literature on distributional properties of the Ising model.

## 1.2 Organization

Section 2 formalizes the setup. Section 3 presents pointwise in  $\beta \in [0, 1]$  distribution theory for the Hajek estimator. Section 4 presents uniform in  $\beta \in [0, 1]$  distribution theory, and discusses an infeasible uniformly valid inference procedure. Section 5 proposes a feasible inference procedure based on resampling methods. Section 6 presents simulation evidence. Section 7 overviews our technical contributions, including Berry-Esseen bounds for Curie-Weiss magnetization with independent multipliers, and Section 8 concludes with open questions and future research directions.

## 2 Setup

We consider a random potential outcome framework under network interference. For each unit  $i \in [n] = \{1, 2, \dots, n\}$ , let  $Y_i(t; \mathbf{t}_{-i})$  denote its random potential outcome when assigned to treatment level  $t \in \{0, 1\}$  while the other units are assigned to treatment levels  $\mathbf{t}_{-i} \in \{0, 1\}^{n-1}$ . The vector of observed random treatment assignments for the  $n$  units is  $\mathbf{T} = (T_i : i \in [n])$ , and  $\mathbf{T}_{-i}$  denotes the associated random treatment assignment vector excluding  $T_i$ . Thus, the observed data is  $(Y_i, T_i : i \in [n])$  with  $Y_i = (1 - T_i)Y_i(0; \mathbf{T}_{-i}) + T_iY_i(1; \mathbf{T}_{-i})$  for each  $i \in [n]$ .

Interference among the  $n$  units is modelled via a latent network characterized by an undirected random graph  $G(\mathbf{V}, \mathbf{E})$  with vertex set  $\mathbf{V} = [n]$  and (random) adjacency matrix  $\mathbf{E} = (E_{ij} : (i, j) \in [n] \times [n]) \in \{0, 1\}^{n \times n}$ . The following assumption restricts this random graph structure.

**Assumption 1** (Network Structure). *The random network  $\mathbf{E}$  satisfies: For all  $1 \leq i \leq j \leq n$  and  $\rho_n \in (0, 1]$ ,  $E_{ii} = 0$ ,  $E_{ij} = E_{ji}$ , and  $E_{ij} = \mathbb{1}(\xi_{ij} \leq \min\{1, \rho_n G(U_i, U_j)\})$ , where  $G : [0, 1]^2 \mapsto \mathbb{R}_+$  is symmetric, continuous and positive on  $[0, 1]^2$ ,  $\mathbf{U} = (U_i : i \in [n])$  are i.i.d.  $\text{Uniform}[0, 1]$  random variables,  $\mathbf{\Xi} = (\xi_{ij} : (i, j) \in [n] \times [n], i < j)$  are i.i.d  $\text{Uniform}[0, 1]$  random variables. Finally,  $\mathbf{U}$  and  $\mathbf{\Xi}$  are independent.*

This assumption corresponds to the *sparse graphon model* of [3]. The parameter  $\rho_n$  controls the sparsity of the network, and will play an important role in our theoretical results. The variable  $U_i$  is a *latent* heterogenous property of the  $i$ th unit, and  $G(U_i, U_j)$  measures similarity between traits of  $U_i$  and  $U_j$ . This allows for a stochastic model for the edge formation.

Building on the underlying random graph structure, the following assumption imposes discipline on the interference entering the outcome equation.

**Assumption 2** (Exchangable Smooth Potential Outcomes Model). *For all  $i \in [n]$ ,  $Y_i(T_i; \mathbf{T}_{-i}) = f_i(T_i; \frac{M_i}{N_i})$  where  $M_i = \sum_{j \neq i} E_{ij} T_j$ ,  $N_i = \sum_{j \neq i} T_j$ , and  $\mathbf{f} = (f_i : i \in [n])$  are i.i.d random functions. In addition, for all  $i \in [n]$  and some integer  $p \geq 4$ ,  $\max_{1 \leq i \leq n} \max_{t \in \{0, 1\}} |\partial_2^{(p)} f_i(t, \cdot)| < C$  for some  $C$  not depending on  $n$  and  $\beta$ . Finally,  $\mathbf{f}$  is independent to  $\mathbf{\Xi}$ .*

This second assumption imposes two main restrictions on the potential outcomes. First, a dimension reduction is assumed via the underlying network structure (Assumption 1), making the potential outcomes for each unit  $i \in [n]$  a function of only their own treatment assignment and the fraction of other treated units among their (connected) peers. Second, the potential outcomes are assumed to be smooth as a function of the fraction of treated peers, thereby ruling out certain types of outcome variables (e.g., binary or similarly limited dependent variable models). Assumption 2 explicitly parametrizes the smoothness level  $p$  because, together with the the sparsity parameter  $\rho_n$  in Assumption 1, it will play an important role in our theoretical results.

To close the causal inference model, the following assumption restricts the treatment assignment distribution. We propose an Ising model from statistical physics [7].

**Assumption 3** (Ising Equiprobable Treatment Assignment). *The treatment assignment mechanism follows a Curie-Weiss distribution:*

$$\mathbb{P}_\beta(\mathbf{T} = \mathbf{t}) = \frac{1}{C_\beta} \exp\left(\frac{\beta}{n} \sum_{i \neq j} (2t_i - 1)(2t_j - 1)\right), \quad (2)$$

where  $\mathbf{t} \in \{0, 1\}^n$ ,  $\beta \in [0, 1]$ , and  $C_\beta$  is determined by the condition  $\sum_{\mathbf{t}} \mathbb{P}_\beta(\mathbf{T} = \mathbf{t}) = 1$ .

This model naturally encodes a class of equiprobable, possibly dependent treatment assignment mechanisms. Assumption 3 implies  $\mathbb{P}_\beta(T_i = 1) = \frac{1}{2}$  for  $i \in [n]$  and all  $\beta \geq 0$ , but allows for correlation in treatment assignment as controlled by  $\beta$ . When  $\beta = 0$ , treatment assignment becomes independent across units, and thus the assignment mechanism reduces to the canonical (equiprobable) randomized allocation. For  $\beta \in [0, 1]$ , the Ising mechanism induces positive pairwise correlations, capturing social interdependence phenomena like peer influence [14] characteristic of observational settings.

We propose a robust inference procedure based on the popular Hajek estimator

$$\hat{\tau}_n = \frac{\sum_{i=1}^n T_i Y_i}{\sum_{i=1}^n T_i} - \frac{\sum_{i=1}^n (1 - T_i) Y_i}{\sum_{i=1}^n (1 - T_i)}. \quad (3)$$

This classical estimator is commonly used in causal inference, both with and without interference. In particular, [12] studied the asymptotic properties of  $\hat{\tau}_n$  when  $\beta = 0$ , under Assumptions 1–3, and showed that

$$\sqrt{n}(\hat{\tau}_n - \tau_n) \rightsquigarrow \mathbf{N}(0, \kappa_2), \quad \kappa_s = \mathbb{E}[(R_i - \mathbb{E}[R_i] + Q_i)^s], \quad (4)$$

where  $\rightsquigarrow$  denotes weak convergence as  $n \rightarrow \infty$ , the standard target is the (conditional) direct average treatment effect given by

$$\tau_n = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[Y_i(1; \mathbf{T}_{-i}) - Y_i(0; \mathbf{T}_{-i}) | f_i(\cdot), \mathbf{E}], \quad (5)$$

and  $R_i = f_i(1, \frac{1}{2}) + f_i(0, \frac{1}{2})$  and  $Q_i = \mathbb{E}\left[\frac{G(U_i, U_j)}{\mathbb{E}[G(U_i, U_j) | U_j]} (f'_j(1, \frac{1}{2}) - f'_j(0, \frac{1}{2})) | U_i\right]$ . The (conditional) direct average treatment effect in (5) is a *predictand*, not an *estimand*, in the sense that it is a random variable that needs not to settle to a non-random probability limit under the assumptions imposed. Consequently, our uncertainty quantification methods can be regarded as prediction intervals for the classical target predictand  $\tau_n$  in the causal inference literature.

### 3 Pointwise Distribution Theory

Our first main result is a Berry-Esseen bound for the Hajek estimator, pointwise in  $\beta \in [0, 1]$ , that is, the degree of treatment assignment interference. We provide a proof sketch in Section 7, with full technical details deferred to the supplementary material.

**Theorem 3.1** (Pointwise Distribution Theory). *Suppose Assumptions 1, 2, and 3 hold. Then,*

$$\sup_{t \in \mathbb{R}} |\mathbb{P}[\widehat{\tau}_n - \tau_n \leq t] - L_n(t; \beta, \kappa_1, \kappa_2)| = O\left(\frac{\log n}{\sqrt{n\rho_n}} + \mathbf{r}_{n,\beta}\right),$$

where  $L_n(\cdot; \beta, \kappa_1, \kappa_2)$  and  $\mathbf{r}_{n,\beta}$  are as follows. Then: (1) High temperature: if  $\beta \in [0, 1)$ ,

$$L_n(t; \beta, \kappa_1, \kappa_2) = \mathbb{P}_\beta \left[ n^{-1/2} \left( \kappa_2 + \kappa_1^2 \frac{\beta}{1-\beta} \right)^{1/2} Z \leq t \right] \quad (6)$$

with  $Z \sim \mathbf{N}(0, 1)$ , and  $\mathbf{r}_{n,\beta} = \sqrt{n \log n} (n\rho_n)^{-\frac{p+1}{2}}$ .

(2) Critical temperature: if  $\beta = 1$ ,

$$L_n(t; \beta, \kappa_1, \kappa_2) = \mathbb{P}_\beta [n^{-1/4} \kappa_1 \mathbf{W}_0 \leq t] \quad (7)$$

with  $\mathbb{P}[\mathbf{W}_0 \leq w] = \frac{\int_{-\infty}^w \exp(-z^4/12) dz}{\int_{-\infty}^{\infty} \exp(-z^4/12) dz}$ ,  $w \in \mathbb{R}$ , and  $\mathbf{r}_{n,\beta} = (\log n)^3 n^{-\frac{1}{4}} + \sqrt[4]{n} \sqrt{\log n} (n\rho_n)^{-\frac{p+1}{2}}$ .

In the high temperature regime ( $\beta \in [0, 1)$ ),  $\sqrt{n}(\widehat{\tau}_n - \tau_n)$  is asymptotically normal with variance  $\kappa_2 + \kappa_1^2 \frac{\beta}{1-\beta}$ . Thus, when  $\beta = 0$ , our result recovers (4), but for  $\beta \in (0, 1)$ , the asymptotic variance is strictly increasing unless  $\kappa_1 = 0$  (i.e., no randomness from the underlying network). In the Critical temperature regime ( $\beta = 1$ ), the limiting distribution is non-Gaussian. The distinct asymptotic behaviors of  $\widehat{\tau}_n$  across these regimes mirror the phase transition phenomena observed in the Ising model's magnetization  $m = \frac{1}{n} \sum_{i=1}^n (2T_i - 1)$ . The first term in the Berry-Esseen bound,  $\log n (n\rho_n)^{-1/2}$ , is not improvable beyond the extra logarithmic factor because  $\log(n)n^{-1/2}$  when  $\rho_n \asymp 1$ . For the second term,  $\mathbf{r}_{n,\beta}$ , the bound depends on the smoothness  $p$  of the potential outcome function and the temperature regime.

Theorem 3.1 highlights key challenges in uncertainty quantification, with unknown quantities  $\kappa_1$  and  $\kappa_2$ , and the unknown regime parameter  $\beta \in [0, 1]$ . Furthermore, [2] established an impossibility result showing that no consistent estimator for  $\beta$  exists in the high-temperature regime. In the following section, we address the estimation of  $\beta$  and the complications arising from the discontinuous transition between Gaussian and non-Gaussian laws.

## 4 Infeasible Robust Inference

This section addresses inference on the treatment effect when the regime parameter  $\beta$  is unknown, but assuming that  $\kappa_1$  and  $\kappa_2$  are known.

### 4.1 Maximum Pseudo-Likelihood Estimator (MPLE) for Temperature

Due to the existence of the normalizing constant  $C_\beta$  in (2), maximum likelihood estimation is not computationally efficient. However, the conditional distribution of  $T_i$  given the rest

of treatments adopts a closed form solution and can be optimized efficiently [17]. Define  $W_i = 2T_i - 1$ ,  $\mathbf{W}_{-i} = \{W_j : j \in [n], j \neq i\}$ , and  $m_i = \frac{1}{n} \sum_{j \neq i} W_j$ . The MPLE for  $\beta$  is

$$\hat{\beta}_n = \arg \max_{\beta \in [0,1]} \sum_{i \in [n]} \log \mathbb{P}_\beta[W_i | \mathbf{W}_{-i}] = \arg \max_{\beta \in [0,1]} \sum_{i \in [n]} -\log \left( \frac{1}{2} W_i \tanh(\beta m_i) + \frac{1}{2} \right).$$

We show in Lemma 6 in the supplementary appendix that the limiting distribution of  $\hat{\beta}_n$  also depends on the regime  $\beta \in [0, 1]$ . For  $\beta \in [0, 1)$ ,  $1 - \hat{\beta}_n \rightsquigarrow (1 - \beta) \max\{(\chi_1^2)^{-1}, 0\}$ , thereby ruling out consistent estimation. For  $\beta = 1$ ,  $\sqrt{n}(\hat{\beta}_n - 1) \rightsquigarrow \min\{W_0^2/3 - 1/W_0^2, 1\}$ , where  $W_0$  is given in Theorem 3.1. For fixed  $n$ , the distribution of  $\hat{\beta}_n - 1$  exhibits the same discontinuity at  $\beta = 1$  as  $\hat{\tau}_n - \tau_n$ , highlighting the need for a distributional approximation that is uniform in  $\beta$  for valid inference across all regimes.

## 4.2 Robust Distribution Theory

We develop valid large sample inference for all values of  $\beta \in [0, 1]$ . From Theorem 3.1, for all  $\beta \in [0, 1)$ , the limiting variance of  $\sqrt{n}(\hat{\tau}_n - \tau_n)$  is  $\kappa_2 + \kappa_1^2 \frac{\beta}{1-\beta}$ . Thus, when  $\kappa_1 \neq 0$ , the asymptotic variance diverges as  $\beta$  approaches the critical value  $\beta = 1$ . In contrast, Theorem 3.1 shows that when  $\beta = 1$  the limiting variance of  $n^{1/4}(\hat{\tau}_n - \tau_n)$  is finite. This discrepancy indicates a lack of uniform validity in the distributional approximations in Theorem 3.1. To address this issue, we establish a uniform distributional approximation based on the drifting sequence  $\beta_n = 1 + \frac{c}{\sqrt{n}}$ . This sequence follows the *knife-edge* rate, ensuring that the law of  $\hat{\tau}_n - \tau_n$  smoothly interpolates between the pointwise distributional approximations indexed by  $\beta \in [0, 1]$ .

**Theorem 4.1** (Robust Distribution Theory). *Suppose Assumptions 1, 2 and 3 hold. Define  $c_{\beta,n} = \sqrt{n}(1 - \beta)$ . Then,*

$$\lim_{n \rightarrow \infty} \sup_{0 \leq \beta \leq 1} \sup_{t \in \mathbb{R}} \left| \mathbb{P}_\beta[\hat{\tau}_n - \tau_n \leq t] - \mathbb{P}_\beta[n^{-\frac{1}{2}} \kappa_2^{\frac{1}{2}} Z + \beta^{\frac{1}{2}} n^{-\frac{1}{4}} \kappa_1 \mathbf{W}_{c_{\beta,n}} \leq t] \right| = 0$$

with  $Z \sim \mathbf{N}(0, 1)$  independent of  $\mathbf{W}_c$ , and  $\mathbb{P}[\mathbf{W}_c \leq w] = \frac{\int_{-\infty}^w \exp(-\frac{x^4}{12} - \frac{cx^2}{2}) dx}{\int_{-\infty}^{\infty} \exp(-\frac{x^4}{12} - \frac{cx^2}{2}) dx}$ ,  $w \in \mathbb{R}$ . Furthermore,

$$\lim_{n \rightarrow \infty} \sup_{0 \leq \beta \leq 1} \sup_{t \in \mathbb{R}} \left| \mathbb{P}_\beta[1 - \hat{\beta}_n \leq t] - \mathbb{P}_\beta[\min\{\max\{\mathbf{T}_{c_{\beta,n},n}^{-2} - \mathbf{T}_{c_{\beta,n},n}^2/(3n), 0\}, 1\} \leq t] \right| = 0$$

where  $\mathbf{T}_{c,n} = Z + n^{\frac{1}{4}} \mathbf{W}_c$ .

Theorem 4.1 establishes that  $H_n(t; \kappa_1, \kappa_2, c_{\beta,n}) = \mathbb{P}_\beta[n^{-\frac{1}{2}} \kappa_2^{\frac{1}{2}} Z + \beta^{\frac{1}{2}} n^{-\frac{1}{4}} \kappa_1 \mathbf{W}_{c_{\beta,n}} \leq t]$  uniformly approximates the distribution of  $\hat{\tau}_n - \tau_n$  in both the high-temperature and critical-temperature regimes. Under the *knife-edge* scaling, the leading term  $n^{-1/2} \kappa_2^{1/2} Z$  becomes negligible, and the typical *knife-edge* representation retains only the second term  $\beta^{1/2} n^{-1/4} \kappa_1 \mathbf{W}_c$ . However, when  $\beta \in [0, 1)$  is fixed and  $c_{\beta,n} = \sqrt{n}(1 - \beta) \rightarrow \infty$ ,  $\mathbf{W}_{c_{\beta,n}}$  approximates  $n^{-1/4} \mathbf{N}(0, (1 - \beta)^{-1})$ , making both terms comparable in order. Consequently, we retain both terms in the distributional approximation. In Lemma 4 in the supplementary, we show that when  $\beta$  is fixed and  $c_{\beta,n} = \sqrt{n}(1 - \beta)$ , we have  $\sup_{t \in \mathbb{R}} |H_n(t; \kappa_1, \kappa_2, c_{\beta,n}) - L_n(t; \kappa_1, \kappa_2, \beta)| \rightarrow 0$ . The same ideas apply to the uniform approximation of  $1 - \hat{\beta}_n$ .

### 4.3 Infeasible Uniform Inference

We can now propose a conservative prediction interval based on the following Bonferroni-correction procedure. In particular, in the first step, a uniform confidence interval for  $\beta$  is constructed under the *knife-edge* approximation, and in the second step, we choose the largest quantile for  $\widehat{\tau}_n - \tau_n$  among all  $\beta$ 's in the confidence interval. The quantile chosen is also based on the *knife-edge* approximation.

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**Algorithm 1:** Infeasible Uniform Inference

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**Input:** Treatments and outcomes  $(T_i, Y_i)_{i \in [n]}$ , MPLE-estimator  $\widehat{\beta}_n$ , an upper bound  $K_n$  such that  $\kappa_2^{\frac{1}{2}} \leq K_n$ , confidence level parameters  $\alpha_1, \alpha_2 \in (0, 1)$ .

**Output:** An  $(1 - \alpha_1 - \alpha_2)$  prediction interval  $\mathcal{C}^\dagger(\alpha_1, \alpha_2)$  for  $\tau_n$ .

Get the maximum pseudo-likelihood estimator  $\widehat{\beta}_n$  of  $\beta$ ;

Define the  $(1 - \alpha_1)$ -confidence region given by  $\mathcal{I}(\alpha_1) = \{\beta \in [0, 1] : 1 - \widehat{\beta}_n \in [\mathbf{q}, \infty)\}$ , where  $\mathbf{q} = \inf\{q : \mathbb{P}[\min\{\max\{\mathbb{T}_{c_{\beta,n},n}^{-2} - \mathbb{T}_{c_{\beta,n},n}^2/(3n), 0\}, 1\} \leq q] \geq \alpha_1\}$ ;

Take  $\mathbf{U} = \sup_{\beta \in \mathcal{I}(\alpha_1)} H_n(1 - \frac{\alpha_2}{2}; K_n, K_n, c_{\beta,n})$ ,  $\mathbf{L} = \inf_{\beta \in \mathcal{I}(\alpha_1)} H_n(\frac{\alpha_2}{2}; K_n, K_n, c_{\beta,n})$ .

**return**  $\mathcal{C}^\dagger(\alpha_1, \alpha_2) = [\widehat{\tau}_n + \mathbf{L}, \widehat{\tau}_n + \mathbf{U}]$ .

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**Theorem 4.2** (Infeasible Uniform Inference). *Suppose Assumptions 1, 2 and 3 hold, and let  $K_n$  be a sequence such that  $\kappa_2^{\frac{1}{2}} \leq K_n$ . Then, the prediction interval given by Algorithm 1 satisfies  $\liminf_{n \rightarrow \infty} \inf_{\beta \in [0,1]} \mathbb{P}_\beta(\tau_n \in \mathcal{C}^\dagger(\alpha_1, \alpha_2)) \geq 1 - \alpha_1 - \alpha_2$ .*

Theorem 4.2 gives a lower bound on the coverage of the proposed confidence region. Algorithm 2 can be implemented without the knowledge of the parameter of the Ising treatment model, but requires knowledge of  $\kappa_1$  and  $\kappa_2$ . A fully feasible implementation is discussed next.

## 5 Implementation

The unknown parameters  $\kappa_1$  and  $\kappa_2$  capture moments of the underlying random graph structure. Building on [13], we propose a resampling method for consistent estimation of those parameters under an additional nonparametric assumption on the outcome equation.

**Assumption 4.** *Suppose  $f_i(\cdot, \cdot) = f(\cdot, \cdot) + \varepsilon_i$ , where  $f(t, \cdot)$  is 4-times continuously differentiable on  $[0, 1]$  for  $t \in \{0, 1\}$ , and  $(\varepsilon_i : 1 \leq i \leq n)$  are i.i.d and independent of  $\mathbf{E}$  and  $\mathbf{T}$ , with  $\mathbb{E}[\varepsilon_i] = 0$  and  $\mathbb{E}[|\varepsilon_i|^{2+\nu}] < \infty$  for some  $\nu > 0$ .*

This assumption allows for nonparametric learning the regression function  $f$ . In Section 4 in the supplementary material, we provide one example of such learner, but here we remain agnostic and thus present high-level conditions. This step aims to find a consistent estimate for both the function  $f$  and its derivative  $\frac{\partial f(\cdot, x)}{\partial x}$ , which can be achieved through the introduction of Assumption 4. We propose the following novel algorithm for estimating  $\kappa_2$  based on resampling methods.

---

**Algorithm 2:** Estimation of  $\kappa_2$ 

---

**Input:** Treatments and outcomes  $(T_i, Y_i)_{i \in [n]}$ , realized graph  $\mathbf{E}$ , non-parametric learner  $\hat{f}$  of  $f$ .

**Output:** An upper bound  $\hat{K}_n$  for  $\kappa_2$ .

Generate a new sample  $(T_i^* : 1 \leq i \leq n)$  with  $\beta = 0$ ;

Take  $M_j^* = \sum_{l \neq j} E_{jl} T_l^*$ ,  $N_j^* = \sum_{l \neq j} E_{jl}$ ,  $M_{j,(i)}^* = \sum_{l \neq i,j} E_{jl} T_l^*$ ,  $N_{j,(i)}^* = \sum_{l \neq i,j} E_{jl}$ ;

Take  $\hat{\varepsilon}_i = Y_i - T_i \hat{f}(1, \frac{\sum_{j \neq i} E_{ij} T_j}{\sum_{j \neq i} T_j}) - (1 - T_i) \hat{f}(0, \frac{\sum_{j \neq i} E_{ij} T_j}{\sum_{j \neq i} T_j})$ ;

Take  $\tau_{(i)}^a = n^{-1} \sum_{j \neq i} 2T_j^* (\hat{f}(1, \frac{M_j^*}{N_j^*}) + \hat{\varepsilon}_j) - 2(1 - T_j^*) (\hat{f}(0, \frac{M_j^*}{N_j^*}) + \hat{\varepsilon}_j)$ , and

$\tau_{(i)}^b = n^{-1} \sum_{j \in [n]} 2T_j^* (\hat{f}(1, \frac{M_{j,(i)}^*}{N_{j,(i)}^*}) + \hat{\varepsilon}_j) - 2(1 - T_i^*) (\hat{f}(0, \frac{M_{j,(i)}^*}{N_{j,(i)}^*}) + \hat{\varepsilon}_j)$ ;

Take  $\bar{\tau}^a = n^{-1} \sum_{i \in [n]} \tau_{(i)}^a$ ,  $\bar{\tau}^b = n^{-1} \sum_{i \in [n]} \tau_{(i)}^b$ , and

$\hat{K}_n = n \sum_{i \in [n]} (\tau_{(i)}^a - \bar{\tau}^a + \tau_{(i)}^b - \bar{\tau}^b)^2$ .

**return**  $\hat{K}_n$ .

---

Our procedure consists of three steps. In step 1, we estimate  $f$  non-parametrically by  $\hat{f}$ . In step 2, we construct *two* types of plug-in and leave-one-out estimator, denoted by  $\{\tau_{(i)}^a\}_{i \in [n]}$  and  $\{\tau_{(i)}^b\}_{i \in [n]}$  respectively.  $\tau_{(i)}^a$  accounts for the randomness from flipping  $i$ -th unit's own treatment.  $\tau_{(i)}^b$  accounts for randomness from flipping  $j$ -th unit's treatment, where  $j$  is a neighbor of  $i$ . In Step 3, we form our final variance estimator using the resampling based treatment effect estimators similar to the i.i.d. case. Formal results on the guarantees given in Lemma 16 in the supplementary material.

---

**Algorithm 3:** Feasible Uniform Inference

---

**Input:** Treatments and outcomes  $(T_i, Y_i)_{i \in [n]}$ , realized graph  $\mathbf{E}$ , non-parametric learner  $\hat{f}$  of  $f$ .

**Output:** A fully data-driven  $(1 - \alpha_1 - \alpha_2)$  prediction interval  $\hat{\mathcal{C}}(\alpha_1, \alpha_2)$  for  $\tau_n$ .

Get  $\hat{K}_n$  from Algorithm 2 using the treatments and outcomes  $(T_i, Y_i)_{i \in [n]}$ , the realized random graph  $\mathbf{E}$ , a non-parametric learner  $\hat{f}$  for  $f$ ;

Get  $\hat{\mathcal{C}}(\alpha_1, \alpha_2)$  from Algorithm 1 given  $(T_i, Y_i)_{i \in [n]}$  and  $\hat{K}_n$ .

**return**  $\hat{\mathcal{C}}(\alpha_1, \alpha_2)$ .

---

**Theorem 5.1** (Feasible Robust Confidence Interval). *Suppose Assumptions 1, 2, 3, and 4 hold. Suppose the non-parametric learner  $\hat{f}$  satisfies  $\hat{f}(\ell, \cdot) \in C_2([0, 1])$ , and  $|\hat{f}(\ell, \pi_*) - f(\ell, \pi_*)| = o_{\mathbb{P}}(1)$ ,  $|\partial_2 \hat{f}(\ell, \pi_*) - \partial_2 f(\ell, \pi_*)| = o_{\mathbb{P}}(1)$ , for  $\ell \in \{0, 1\}$ . If  $n\rho_n^3 \rightarrow \infty$ , then the prediction interval given by Algorithm 3 satisfies*

$$\liminf_{n \rightarrow \infty} \sup_{\beta \in [0, 1]} \mathbb{P}_{\beta}[\tau_n \in \hat{\mathcal{C}}(\alpha_1, \alpha_2)] \geq 1 - \alpha_1 - \alpha_2.$$

## 6 Simulations

We study the finite sample performance of our robust inference procedure. Take  $(U_i : 1 \leq i \leq n)$  i.i.d Uniform( $[0, 1]$ )-distributed, graph function  $G(\cdot, \cdot) \equiv 0.5$  and density  $\rho_n = 0.5$ . The

Ising-treatments satisfy Assumption 3 with various  $n$  and  $\beta$ .  $Y_i$  has data generating process  $Y_i = \mathbb{1}(T_i = 1)f(1, \frac{M_i}{N_i}) + \mathbb{1}(T_i = 0)f(0, \frac{M_i}{N_i}) + \varepsilon_i$ , with  $f(x_1, x_2) = x_1^2 + x_1(x_2 + 1)^2$ ,  $(x_1, x_2) \in \mathbb{R}^2$  and  $(\varepsilon_i : 1 \leq i \leq n)$  are i.i.d  $N(0, 0.05)$  noise terms independent to  $((U_i, T_i) : 1 \leq i \leq n)$ . The Monte-Carlo simulations are repeated with 5000 iterations and look at the  $1 - \alpha$  confidence interval with  $\alpha = 0.1$ .

Figure 1 (a) and (b) demonstrate the empirical coverage and interval length against  $\beta$ , while fixing  $n = 500$ . To compare multiple methods, **conserv** stands for Algorithm 3, " $\beta = 0$ " stands for using the formula from Theorem 3.1, **Oracle** stands for using the law  $n^{-1/2}\widehat{\kappa}_2^{1/2}\mathbf{Z} + n^{-1/4}\widehat{\kappa}_1\mathbf{W}_{c_{\beta,n}}$  from Theorem 4.1 with  $c_{\beta,n} = \sqrt{n}(1 - \beta)$  assumed to be known, and **Onestep** stands for Algorithm 1 but taking the first step confidence interval  $\mathbb{I}(\alpha_1)$  to be the full range  $[0, 1]$  instead. For interval length, **Simulated** stands for the true interval length from Monte-Carlo simulations. **Conservative** and **Onestep** remain conservative except when  $\beta$  is close to 1, due to the second step in Algorithm 1 taking maximum quantile from  $\beta \in \mathbb{I}(\alpha_1)$ ; **Oracle** has empirical coverage close to  $1 - \alpha$  and interval length close to the true interval length from Monte-Carlo simulation; the approach of plugging in  $\beta = 0$  becomes invalid as  $\beta$  deviates from zero. Figure 1 (c) and (d) demonstrate log-log plots of interval length against sample size, fixing  $\beta = 0$ . While the Monte-Carlo interval length **Simulated** interval length  $\propto n^{-0.52}$ , consistent with the  $\sqrt{n}$ -convergence with  $\beta = 0$ , **Conserv** has interval length  $\propto n^{-0.34}$ , an effect of taking the maximum quantile among  $\beta \in \mathbb{I}(\alpha_1)$ .

## 7 Main Technical Contribution

This section reports the main novel technical result in our paper: a Berry-Esseen distributional approximation for Curie-Weiss magnetization with independent multipliers. This section is self-contained, but omitted details are given in the supplemental appendix.

**Lemma 7.1** (Ising Berry-Esseen Bound). *For  $\beta \geq 0$ , suppose  $\mathbb{P}[\mathbf{W} = \mathbf{w}] \propto \exp(\frac{\beta}{n} \sum_{i \neq j} w_i w_j)$ , where  $\mathbf{W} = (W_1, \dots, W_n)^\top$ ,  $\mathbf{w} = (w_1, \dots, w_n)^\top \in \{-1, 1\}^n$ , and  $(X_1, \dots, X_n)$  are i.i.d. with  $\mathbb{E}[|X_i|^3] < \infty$ , and independent of  $\mathbf{W}$ . Then:*

(1) Fix  $\beta \in [0, 1]$ , then  $\sup_{t \in \mathbb{R}} |\mathbb{P}(\frac{1}{n} \sum_{i=1}^n X_i W_i \leq t) - L_n(t; (\mathbb{E}[X_i], \mathbb{E}[X_i^2]), \beta)| = O(\mathbf{r}_{n,\beta})$ , where  $\mathbf{r}_{n,\beta} = n^{-1/2}$  for  $\beta \in [0, 1]$ ,  $\mathbf{r}_{n,\beta} = n^{-1/2}(\log n)^3$  for  $\beta = 1$ , where  $L_n$  is given in Theorem 3.1.

(2)  $\sup_{\beta \in [0, 1]} \sup_{t \in \mathbb{R}} |\mathbb{P}(\frac{1}{n} \sum_{i=1}^n X_i W_i \leq t) - H_n(t; \mathbb{E}[X_i], \mathbb{E}[X_i^2], c_{\beta,n})| = O(n^{-1/2}(\log n)^3)$ , where  $c_{\beta,n} = \sqrt{n}(\beta - 1)$ , and  $H_n$  is given immediately after Theorem 4.1.

These result generalize the Berry-Esseen bounds for Curie-Weiss magnetization  $\frac{1}{n} \sum_{i=1}^n W_i$  with multipliers set to  $X_i = 1$  for  $i \in [n]$  obtained by [5] and [6]. Our generalized result differs from theirs only in a logarithmic term, allowing for fairly general weights with third moment bounded.

### 7.1 Proof Sketch of Lemma 7.1

The magnetization  $n^{-1} \sum_{i=1}^n W_i$  has been studied using Stein's method [6, 4]. Due to the multipliers, the Stein's method can not be directly applied for  $n^{-1} \sum_{i=1}^n X_i W_i$ . We use a novel strategy based on the following de Finetti's lemma.

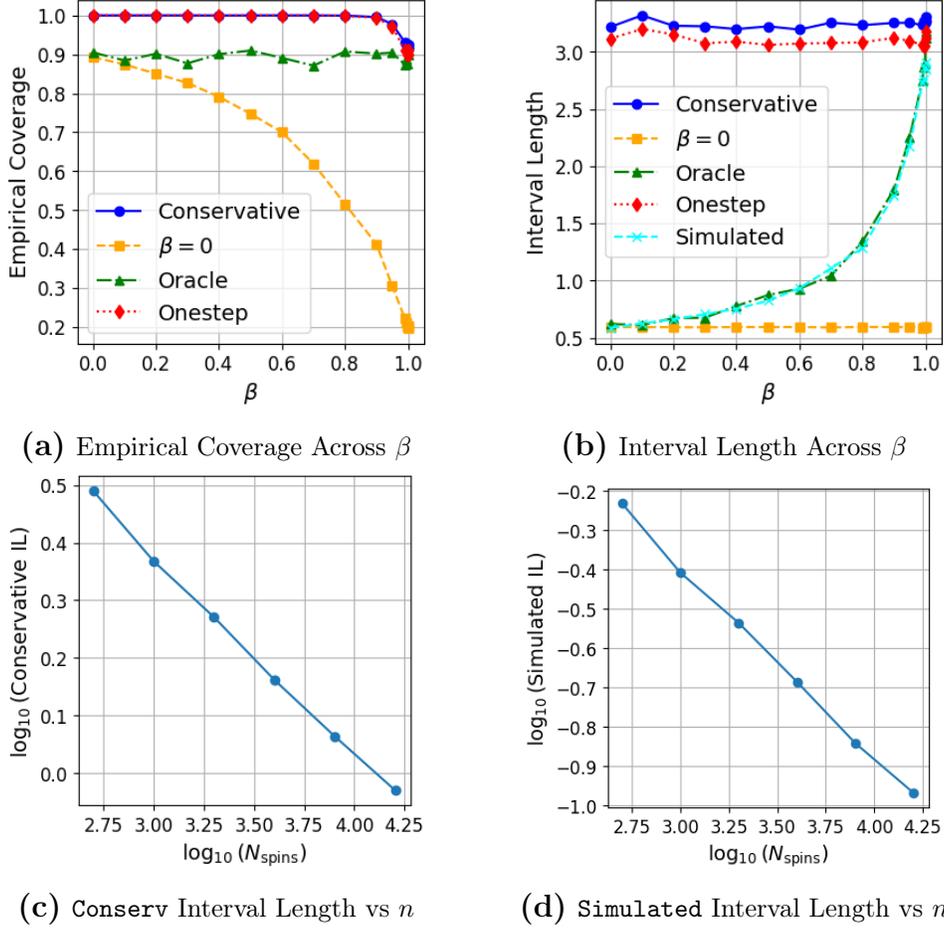


Figure 1: (a) and (b) are empirical coverages and interval lengths of four methods across  $\beta \in [0, 1]$ : **Conservative** and **Onestep** remain conservative except when  $\beta$  is close to 1; **Oracle** has empirical coverage close to  $1 - \alpha$  and interval length close to the true interval length from Monte-Carlo simulation; the approach of plugging in  $\beta = 0$  becomes invalid as  $\beta$  deviates from zero. (c) shows **Conserv** interval length  $\propto n^{-0.34}$ . (d) shows **Simulated** interval length  $\propto n^{-0.52}$ .

*de Finetti's Lemma.* There exists a latent variable  $\mathbf{U}_n$  such that  $W_1, \dots, W_n$  are i.i.d condition on  $\mathbf{U}_n$ . Moreover, the density of  $\mathbf{U}_n$  satisfies  $f_{\mathbf{U}_n}(u) \propto \exp(-\frac{1}{2}u^2 + n \log \cosh(\sqrt{\beta/nu}))$ ,  $u \in \mathbb{R}$ .

We provide a proof sketch of Lemma 7.1 (2). Rigorous proofs for the other regimes given in Section 1 of the supplementary material. Denote by  $\mathcal{C}$  an absolute constant,  $K$  a constant that only depends on the distribution of  $X_i$ , and  $O(\cdot)$  is by an absolute constant. Throughout, take  $c = \sqrt{n}(\beta - 1)$ .

**Step 1: Conditional Berry-Esseen.**  $W_i$ 's are i.i.d condition on  $\mathbf{U}_n$  with  $e(\mathbf{U}_n) = \mathbb{E}[X_i W_i | \mathbf{U}_n] = \mathbb{E}[X_i] \tanh(\sqrt{\beta/n} \mathbf{U}_n)$ , and  $v(\mathbf{U}_n) = \mathbb{V}[X_i W_i | \mathbf{U}_n] = \mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2 \tanh^2(\sqrt{\beta/n} \mathbf{U}_n)$ . Apply Berry-Esseen Theorem conditional on  $\mathbf{U}_n$ , and take  $\mathbf{Z} \sim \mathcal{N}(0, 1)$  independent to  $\mathbf{U}_n$ ,

$$\sup_{t \in \mathbb{R}} |\mathbb{P}(\frac{1}{n} \sum_{i=1}^n X_i W_i \leq t | \mathbf{U}_n) - \mathbb{P}(\sqrt{v(\mathbf{U}_n)} \mathbf{Z} + \sqrt{n} e(\mathbf{U}_n) \leq t | \mathbf{U}_n)| \leq \mathcal{C} \mathbb{E}[|X_i|^3] v(\mathbf{U}_n) n^{-1/2}.$$

Lemma 2 in the supplementary material shows  $\|\mathbf{U}_n\|_{\psi_2} \leq \mathcal{C} n^{1/4}$ , hence by concentration

arguments,  $\sup_{t \in \mathbb{R}} |\mathbb{P}(\frac{1}{n} \sum_{i=1}^n X_i W_i \leq t) - \mathbb{P}(\sqrt{v(\mathbf{U}_n)} \mathbf{Z} + \sqrt{n} e(\mathbf{U}_n) \leq t)| \leq K n^{-1/2}$ .

**Step 2: Non-Normal Approximation for  $n^{-1/4} \mathbf{U}_n$ .** Consider  $\mathbf{W}_n = n^{-1/4} \mathbf{U}_n$ . By a change of variable from  $\mathbf{U}_n$  and Taylor expand what is inside the exponent, we show  $\mathbf{W}_n$  has density satisfying

$$f_{\mathbf{W}_n}(w) \propto \exp\left(-\frac{c}{2} w^2 - \frac{\beta_n^2}{12} w^4 + g(w) \beta_n^3 n^{-\frac{1}{2}} w^6\right),$$

where  $g$  is a bounded smooth function. We show based on sub-Gaussianity of  $\mathbf{W}_n$ , with an upper bound of sub-Gaussian norm not depending on  $\beta$ , that the sixth order term is negligible and  $\sup_{t \in \mathbb{R}} |\mathbb{P}(\mathbf{W}_n \leq t) - \mathbb{P}(\mathbf{W} \leq t)| = O((\log n)^3 n^{-1/2})$ , where  $\mathbf{W}$  has density proportional to  $\exp(-\frac{c}{2} w^2 - \frac{\beta_n^2}{12} w^4)$ .

**Step 3: Concentration Arguments.** Since  $\mathbf{Z}$  is independent to  $(\mathbf{U}_n, \mathbf{W}_n)$ , we use data processing inequality and the previous two steps to show  $\frac{1}{n} \sum_{i=1}^n X_i W_i$  is close to  $n^{-\frac{1}{4}} v(n^{\frac{1}{4}} \mathbf{W}_c)^{\frac{1}{2}} \mathbf{Z} + n^{\frac{1}{4}} e(n^{\frac{1}{4}} \mathbf{W}_c)$ . Lemma 2 in the supplementary appendix imply  $\|\mathbf{W}\|_{\psi_2} \leq K$ . By Taylor expanding  $e(\cdot)$  and  $v(\cdot)$  at 0, we show  $n^{1/4} e(\mathbf{U}_n)$  is close to  $\mathbb{E}[X_i] \mathbf{W}$  and  $n^{-1/4} \sqrt{v(\mathbf{U}_n)} \mathbf{Z}$  is close to  $n^{-\frac{1}{4}} v(n^{\frac{1}{4}} \mathbf{W})^{\frac{1}{2}} \mathbf{Z}$ .

## 8 Discussion

This section discusses related results and future research directions.

### 8.1 Low Temperature Regime

The low temperature regime corresponds to  $\beta > 1$ , which was excluded from the main results presented. In this case the Hajek estimator converges to a different (conditional) direct treatment effect that also depends on which side of the half line  $\text{sgn}(m) = \text{sgn}(\frac{2}{n} \sum_{i=1}^n T_i - 1)$  lies on, due to the convergence of  $\frac{M_i}{N_i}$  to a two-point distribution depending on  $\text{sgn}(m)$ . Define

$$\tau_{n,\ell} = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[Y_i(1; \mathbf{T}_{-i}) - Y_i(0; \mathbf{T}_{-i}) | f_i(\cdot), \mathbf{E}, \text{sgn}(m) = \ell], \quad \ell \in \{-, +\},$$

which is a new causal predictand in the context of our causal inference model with interference. In the supplemental appendix, and under the assumptions imposed in the paper, we show that

$$\sup_{t \in \mathbb{R}} \max_{\ell \in \{-, +\}} |\mathbb{P}(\widehat{\tau}_n - \tau_{n,\ell} \leq t | \text{sgn}(m) = \ell) - L_n(t; \beta, \kappa_{1,\ell}, \kappa_{2,\ell})| = O\left(\sqrt{\frac{n \log n}{(n \rho_n)^{p+1}}} + \frac{\log n}{\sqrt{n \rho_n}}\right),$$

where  $\kappa_{s,\ell} = \mathbb{E}[(R_{i,\ell} + Q_{i,\ell})^s]$  with  $R_{i,\ell} = f_i(1, \pi_\ell) - \mathbb{E}[f_i(1, \pi_\ell)] + f_i(0, \pi_\ell) - \mathbb{E}[f_i(0, \pi_\ell)]$  and  $Q_{i,\ell} = \mathbb{E}\left[\frac{G(U_i, U_j)}{\mathbb{E}[G(U_i, U_j) | U_j]} (f_j'(1, \pi_\ell) - f_j'(0, \pi_\ell)) | U_i\right]$ , and

$$L_n(t; \beta, \kappa_1, \kappa_2) = \mathbb{P}\left(n^{-1/2} \left(\kappa_2(1 - \pi_*^2) + \kappa_1^2 \frac{\beta(1 - \pi_*^2)}{1 - \beta(1 - \pi_*^2)}\right)^{1/2} \mathbf{Z} \leq t\right)$$

with  $Z \sim \mathbf{N}(0, 1)$  independent of  $m$ ,  $\pi_*$  the positive root of  $x = \tanh(\beta x)$ , and  $\pi_+ = \frac{1}{2} + \frac{1}{2}\pi_*$ ,  $\pi_- = \frac{1}{2} - \frac{1}{2}\pi_*$ . Inference for the conditional estimand is left for future works, with a challenge in a discontinuity in the estimand as we move from the critical regime to the low temperature regime.

## 8.2 Generalized Ising Model

In this work we assumed treatments are dependent through a fully connected graph. It is also of interest to study settings where the graph underlying treatment assignment has a block structure, or depends on unit-level properties. In the structured Ising setting, we might also consider estimation and inference for the block level or heterogenous direct average treatment effect.

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# Supplementary Material to “Treatment Network Effect Estimation under Dependence”

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## Abstract

This Supplemental Material contains general theoretical results encompassing those discussed in the main paper, includes proofs of those general results, and discusses additional methodological and technical results.

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## SA-1 Notation

For a sequence of real-valued random variables  $X_n$ , we say  $X_n = O_{\psi_p}(r_n)$  if there exists  $N \in \mathbb{N}$  and  $M > 0$  such that  $\|X_n\|_{\psi_p} \leq Mr_n$  for all  $n \geq N$ , where  $\|\cdot\|_{\psi_p}$  is the Orlicz norm w.r.p  $\psi_p(x) = \exp(x^p) - 1$ . We say  $X_n = O_{\psi_p,tc}(r_n)$ ,  $tc$  stands for tail control, if there exists  $N \in \mathbb{N}$  and  $M > 0$  such that for all  $n \geq N$  and  $t > 0$ ,  $\mathbb{P}(|X_n| \geq t) \leq 2n \exp(-(t/(Mr_n))^p) + Mn^{-1/2}$ .

## SA-2 Berry-Esseen Results for Curie-Weiss magnetization with Independent Multipliers

For  $\beta \geq 0$ , the *Curie-Weiss model of ferromagnetic interaction* at inverse temperature  $\beta$  and zero external field is given by the following Gibbs measure on  $\{-1, +1\}^n$ :

$$P_\beta(\mathbf{w}) = \frac{1}{Z_\beta} \exp\left(\frac{\beta}{n} \sum_{i < j} w_i w_j\right), \quad \mathbf{w} = (w_1, \dots, w_n) \in \{-1, 1\}^n, \quad (\text{SA-1})$$

where  $Z_\beta$  is the normalizing constant.

Suppose  $\mathbf{W} = (W_1, \dots, W_n)$  is a random vector with law  $P_\beta$ . Then  $\mathbb{E}[W_i] = 0$  and  $m = n^{-1} \sum_{i=1}^n W_i$ . The Curie-Weiss model has a phase transition phenomena between regimes. The case  $0 \leq \beta \leq 1$  is called the *high temperature* regime, where  $m$  concentrates around 0. The case  $\beta > 1$  is called the *low temperature* regime, where  $m$  concentrates on the set  $\{-\pi_*, \pi_*\}$ ,  $\pi_*$  being the unique positive solution to  $x = \tanh(\beta x)$ . The case  $\beta = 1$  is called the *critical temperature* regime.

Suppose  $\mathbf{X} = (X_1, \dots, X_n)$  has i.i.d components such that  $\mathbb{E}[|X_1|^3] < \infty$  independent to  $\mathbf{W}$ . The goal is to study the limiting distribution and the rate of convergence for

$$g_n = n^{-1} \sum_{i=1}^n W_i X_i.$$

The magnetization  $n^{-1} \sum_{i=1}^n W_i$  has been studied using Stein's method [5], [3]. Due to the multipliers, the Stein's method can not be directly applied for  $g_n$ . We use a novel strategy based on the following de Finetti's lemma to show Berry Esseen results.

**Lemma SA-1** (de Finetti's Lemma). *There exists a latent variable  $U_n$  with density*

$$f_{U_n}(u) = I_{U_n}^{-1} \exp\left(-\frac{1}{2}u^2 + n \log \cosh\left(\sqrt{\frac{\beta}{n}}u\right)\right),$$

where  $I_{U_n} = \int_{-\infty}^{\infty} \exp(-\frac{1}{2}u^2 + n \log \cosh(\sqrt{\frac{\beta}{n}}u)) du$ , such that  $W_1, \dots, W_n$  are i.i.d condition on  $U_n$ .

**Lemma SA-2.** *Take  $U_n$  to be a random variable with density function  $f_{U_n}(u) = I_{U_n}^{-1} \exp(-\frac{1}{2}u^2 + n \log \cosh(\sqrt{\frac{\beta}{n}}u + h))$  where  $I_{U_n} = \int_{-\infty}^{\infty} \exp(-\frac{1}{2}u^2 + n \log \cosh(\sqrt{\beta/n}u + h)) du$ . Take  $W_n = n^{-\frac{1}{4}}U_n$ . Then*

1. *High-temperature case: Suppose  $h \neq 0$  or  $h = 0, \beta < 1$ . Then  $\|U_n - \mathbb{E}[U_n]\|_{\psi_2} \lesssim 1$ .*
2. *Critical-temperature case: Suppose  $h = 0$  and  $\beta = 1$ . Then  $\|U_n\|_{\psi_2} \lesssim n^{1/4}$ .*

3. *Low-temperature case:* Suppose  $h = 0$  and  $\beta > 1$ . Then condition on  $\mathbf{U}_n \in \mathcal{C}_l$ ,  $\|\mathbf{U}_n - \mathbb{E}[\mathbf{U}_n | \mathbf{U}_n \in \mathcal{C}_l]\|_{\psi_2} \lesssim 1$ .
4. *Drifting sequence case:* Suppose  $h = 0$ ,  $\beta = 1 - cn^{-\frac{1}{2}}$ ,  $c \in \mathbb{R}^+$ . Then  $\|\mathbf{U}_n\|_{\psi_2} \leq \mathfrak{C}n^{1/4}$  for large enough  $n$  with  $\mathfrak{C}$  not depending on  $\beta$ .

Fix  $\beta > 0$ . We characterize the limiting distribution of  $n^{-1} \sum_{i=1}^n W_i X_i$  and the rate of convergence as  $n \rightarrow \infty$  in the following lemma. In particular, we will see that the limiting distribution changes from a Gaussian distribution under high temperature, to a non-Gaussian distribution under critical temperature, to a Gaussian mixture under low temperature.

**Lemma SA-3** (Fixed Temperature Berry-Esseen). *Then*

1. When  $\beta < 1$ ,

$$\sup_{t \in \mathbb{R}} |\mathbb{P}(n^{\frac{1}{2}} (\mathbb{E}[X_i^2] + \mathbb{E}[X_i]^2 \frac{\beta}{1-\beta})^{-\frac{1}{2}} g_n \leq t) - \Phi_{N(0,1)}(t)| = O(n^{-\frac{1}{2}}).$$

2. When  $\beta = 1$ , denote  $F_0(t) = \frac{\int_{-\infty}^t \exp(-z^4/12) dz}{\int_{-\infty}^{\infty} \exp(-z^4/12) dz}$ ,  $t \in \mathbb{R}$ , then

$$\sup_{t \in \mathbb{R}} |\mathbb{P}(n^{\frac{1}{4}} \mathbb{E}[X_i]^{-1} g_n \leq t) - F_0(t)| = O((\log n)^3 n^{-\frac{1}{2}}).$$

3. When  $\beta > 1$ , denote  $g_{n,\ell} = \frac{1}{n} \sum_{i=1}^n X_i (W_i - \pi_\ell)$ ,  $\mathcal{C}_+ = [0, \infty)$  and  $\mathcal{C}_- = (-\infty, 0)$ , then

$$\begin{aligned} \sup_{t \in \mathbb{R}} |\mathbb{P}(n^{\frac{1}{2}} \left( \mathbb{E}[X_i^2] (1 - \pi_\ell^2) + \mathbb{E}[X_i]^2 \frac{\beta(1 - \pi_\ell^2)}{1 - \beta(1 - \pi_\ell^2)} \right)^{-\frac{1}{2}} g_{n,\ell} \leq t | m \in \mathcal{C}_\ell) - \Phi_{N(0,1)}(t)| \\ = O(n^{-\frac{1}{2}}), \quad t \in \{-, +\}. \end{aligned}$$

**Lemma SA-4** (Size-Dependent Temperature Berry-Esseen). *Suppose  $Z$  is a standard Gaussian random variable. (1) Suppose  $\beta_n = 1 + cn^{-\frac{1}{2}}$ , where  $c < 0$  does not depend on  $n$ . Then*

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}(n^{\frac{1}{4}} g_n \leq t) - \mathbb{P}(n^{-\frac{1}{4}} \mathbb{E}[X_i^2]^{\frac{1}{2}} Z + \beta_n^{\frac{1}{2}} \mathbb{E}[X_i] W_c \leq t) \right| = O((\log n)^3 n^{-\frac{1}{2}}),$$

where  $O(\cdot)$  is up to a universal constant.

- (2) Suppose  $\beta_n = 1 + cn^{-\frac{1}{2}}$ , where  $c > 0$  does not depend on  $n$ . Then

$$\begin{aligned} \sup_{c \in \mathbb{R}^+} \sup_{t \in \mathbb{R}} \left| \mathbb{P}(n^{\frac{1}{4}} g_n \leq t | m \in \mathcal{I}_{c,\ell}) - \mathbb{P}(n^{-\frac{1}{4}} \mathbb{E}[X_i^2]^{\frac{1}{2}} Z + \beta_n^{\frac{1}{2}} \mathbb{E}[X_i] W_{c,n} \leq t | W_{c,n} \in \mathcal{I}_{c,\ell}) \right| \\ = O((\log n)^3 n^{-\frac{1}{2}}), \end{aligned}$$

with  $\mathcal{I}_{c,n,-} = (-\infty, K_{c,n,-})$  and  $\mathcal{I}_{c,n,+} = (K_{c,n,+}, \infty)$  such that  $\mathbb{E}[W_{c,n} | W_{c,n} \in \mathcal{I}_{c,n,\ell}] = w_{c,n,\ell}$  for  $\ell \in \{-, +\}$ .

**Lemma SA-5** ( $\sqrt{n}$ -sequence is knife-edge). (1) Suppose  $|\beta_n - 1| = o(n^{-\frac{1}{2}})$ , then

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}(n^{\frac{1}{4}} g_n \leq t) - \mathbb{P}(\mathbb{E}[X_i] W_0 \leq t) \right| = o(1).$$

(2) Suppose  $1 - \beta_n \gg n^{-\frac{1}{2}}$ , then

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}(\mathbb{V}[g_n]^{-\frac{1}{2}} g_n \leq t) - \Phi(t) \right| = o(1).$$

(3) Suppose  $\beta_n - 1 \gg n^{-\frac{1}{2}}$ , then for  $\ell \in \{-, +\}$ ,

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}\left(\mathbb{V}[g_n | m \in \mathcal{I}_\ell]^{-\frac{1}{2}} (g_n - \mathbb{E}[g_n | m \in \mathcal{I}_\ell]) \leq t\right) - \Phi(t) \right| = o(1),$$

where  $\mathcal{I}_+ = [0, \infty)$  and  $\mathcal{I}_- = (-\infty, 0)$ .

### SA-3 Pseudo-Likelihood Estimator for Curie-Weiss Regimes

**Lemma SA-1** (No Consistent Variance Estimator). *Suppose Assumptions 1,2,3 hold. Then there is no consistent estimator of  $n\mathbb{V}[\hat{\tau}_n - \tau_n]$ .*

The pseudo-likelihood estimator for Curie-Weiss regime with no external field is given by

$$\begin{aligned} \hat{\beta} &= \arg \max_{\beta} \sum_{i \in [n]} \log \mathbb{P}_{\beta}(W_i | W_{-i}) \\ &= \arg \max_{\beta} \sum_{i \in [n]} -\log \left( \frac{W_i \tanh(\beta n^{-1} \sum_{j \neq i} W_j) + 1}{2} \right). \end{aligned}$$

**Lemma SA-2** (Fixed Temperature Distribution Approximation). (1) If  $\beta \in [0, 1)$ , then

$$\hat{\beta} \xrightarrow{d} \max \left\{ 1 - \frac{1 - \beta}{\chi^2(1)}, 0 \right\}.$$

(2) If  $\beta = 1$ , then

$$n^{\frac{1}{2}}(1 - \hat{\beta}) \xrightarrow{d} \max \left\{ \frac{1}{W_0^2} - \frac{W_0^2}{3}, 0 \right\}.$$

(3) If  $\beta > 1$ , we define an unrestricted pseud-likelihood estimator,

$$\hat{\beta}_{UR} = \arg \max_{\beta \in \mathbb{R}} \log \mathbb{P}_{\beta}(W_i | \mathbf{W}_{-i}) = \sum_{i \in [n]} -\log \left( \frac{1}{2} W_i \tanh(\beta m_i) + \frac{1}{2} \right).$$

Then

$$\sup_{t \in \mathbb{R}} |\mathbb{P}(n^{1/2}(\hat{\beta}_{UR} - \beta) \leq t | m \in \mathcal{I}_\ell) - \mathbb{P}\left(\left(\frac{1 - \beta(1 - \pi_\ell^2)}{1 - \pi_\ell^2}\right)^{1/2} \mathbf{Z} \leq t\right)| = o(1).$$

**Lemma SA-3** (Drifting Temperature Distribution Approximation). *For any  $\beta \in [0, 1]$ , define  $c_{\beta,n} = \sqrt{n}(1 - \beta)$ , and suppose*

$$\mathbb{P}(z_{\beta,n} \leq t) = \mathbb{P}(\mathbf{Z} + n^{\frac{1}{4}} \mathbf{W}_{c_{\beta,n}} \leq t), \quad t \in \mathbb{R}.$$

then

$$\sup_{\beta \in [0,1]} \sup_{t \in \mathbb{R}} |\mathbb{P}(1 - \hat{\beta} \leq t) - \mathbb{P}(\min\{\max\{z_{\beta,n}^{-2} - \frac{1}{3n} z_{\beta,n}^2, 0\}, 1\} \leq t)| = o(1).$$

## SA-4 Stochastic Linearization

Throughout this section, we prove under a more generic setting. We assume  $W_i = 2T_i - 1$ , and  $(W_i)_{i \in [n]}$  satisfies a Curie-Weiss model with a possibly non-zero external field, that is,

**Assumption 1** (Curie-Weiss). *Suppose  $\mathbf{W} = (W_i)_{1 \leq i \leq n}$  are such that for some  $C_{\beta, h} \in \mathbb{R}$ ,*

$$\mathbb{P}(\mathbf{W} = \mathbf{w}) = C_{\beta, h}^{-1} \exp \left( \frac{\beta}{n} \sum_{1 \leq i < j \leq n} W_i W_j + h \sum_{i=1}^n W_i \right),$$

where  $C_{\beta, h}$  is a normalizing constant.

Moreover, for the ease of proof, we let  $g_i$  to be the function such that

$$g_i(x, y) = f_i\left(\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y + \frac{1}{2}\right), \quad x \in \{-1, 1\}, y \in [-1, 1].$$

We denote  $M_i = \sum_{j \neq i} E_{ij} W_j$ ,  $N_i = \sum_{j \neq i} E_{ij}$ . Then

$$g_i(T_i, \mathbf{T}_{-i}) = f_i\left(T_i, \frac{\sum_{j \neq i} E_{ij} T_j}{\sum_{j \neq i} E_{ij}}\right) = g_i\left(W_i, \frac{M_i}{N_i}\right).$$

Define  $\pi = \mathbb{E}[W_i]$ ,  $m = n^{-1} \sum_{i=1}^n W_i$  and for  $1 \leq i \leq n$ ,  $m_i = n^{-1} \sum_{j \neq i} W_j$ . Define the following rates that will be used in the convergence analysis:

$$\mathbf{a}_{\beta, h} = \begin{cases} n^{1/2}, & \text{if } \beta \neq 1 \text{ or } \beta = 1, h \neq 0, \\ n^{3/4}, & \text{if } \beta = 1, h = 0, \end{cases} \quad \mathbf{r}_{\beta, h} = \begin{cases} n^{1/2}, & \text{if } \beta \neq 1 \text{ or } \beta = 1, h \neq 0, \\ n^{1/4}, & \text{if } \beta = 1, h = 0. \end{cases}$$

and

$$\mathbf{p}_{\beta, h} = \begin{cases} 1/2, & \text{if } \beta \neq 1 \text{ or } \beta = 1, h \neq 0, \\ 1/4, & \text{if } \beta = 1, h = 0, \end{cases} \quad \psi_{\beta, h}(x) = \begin{cases} \exp(x^2) - 1, & \text{if } \beta \neq 1 \text{ or } \beta = 1, h \neq 0, \\ \exp(x^4) - 1, & \text{if } \beta = 1, h = 0. \end{cases}$$

### SA-4.1 The Unbiased Estimator

Denote  $p_i = \mathbb{P}(W_i = 1; \mathbf{W}_{-i}) = (\exp(-2\beta m_i - 2h) + 1)^{-1}$ . We propose an unbiased estimator given by

$$\hat{\tau}_{n, \text{UB}} = \frac{1}{n} \sum_{i=1}^n \left[ \frac{T_i Y_i}{p_i} - \frac{(1 - T_i) Y_i}{1 - p_i} \right].$$

**Lemma SA-1** (Unbiased Estimator).  *$\hat{\tau}_{n, \text{UB}}$  is an unbiased estimator for  $\tau_n$  in the sense that,*

$$\mathbb{E}[\hat{\tau}_{n, \text{UB}} | \mathbf{E}, (f_i)_{i \in [n]}] = \tau_n.$$

We will show the followings have weak limits:

$$n^{-\mathbf{a}_{\beta, h}} \sum_{i=1}^n \left[ \frac{T_i Y_i}{p_i} - \frac{(1 - T_i) Y_i}{1 - p_i} - \tau_n \right].$$

W.l.o.g, we analyse the error for treated data, the error for control data follows in the same way. First, decompose by

$$\begin{aligned} n^{-\mathbf{a}_{\beta,h}} \sum_{i=1}^n \left[ \frac{T_i Y_i}{p_i} - \frac{(1-T_i)Y_i}{1-p_i} \right] &= \Delta_1 + \Delta_2, \\ \Delta_1 &= n^{-\mathbf{a}_{\beta,h}} \sum_{i=1}^n \left[ \frac{T_i}{p_i} Y_i(1, \pi) - \frac{1-T_i}{1-p_i} g_i(-1, \pi) \right], \\ \Delta_2 &= n^{-\mathbf{a}_{\beta,h}} \sum_{i=1}^n \left[ \frac{T_i}{p_i} \left( g_i \left( 1, \frac{M_i}{N_i} \right) - g_i(1, \pi) \right) - \frac{1-T_i}{1-p_i} \left( g_i \left( -1, \frac{M_i}{N_i} \right) - g_i(-1, \pi) \right) \right]. \end{aligned}$$

**Lemma SA-2.** *Suppose Assumption 1,2, and 3 hold. Then*

$$\begin{aligned} \Delta_1 - \mathbb{E}[\Delta_1 | \mathbf{E}, (f_i)_{i \in [n]}] &= n^{-\mathbf{a}_{\beta,h}} \sum_{i=1}^n \left( \frac{g_i(1, \pi)}{1+\pi} + \frac{g_i(-1, \pi)}{1-\pi} - \beta \mathbf{d} \right) (W_i - \pi) \\ &\quad + O_{\psi_2, tc}(\sqrt{\log n n^{-\mathbf{r}_{\beta,h}}}), \end{aligned}$$

where  $\mathbf{d} = (1-\pi)\mathbb{E}[g_i(1, \pi)] + (1+\pi)\mathbb{E}[g_i(-1, \pi)]$ .

Now consider  $\Delta_2$ . Since  $\frac{T_i}{p_i} = \frac{T_i - p_i}{p_i} + 1$ , we have the decomposition,

$$\Delta_2 = n^{-\mathbf{a}_{\beta,h}} \sum_{i=1}^n \frac{T_i}{p_i} \left[ g_i \left( 1, \frac{M_i}{N_i} \right) - g_i(1, \pi) \right] = \Delta_{2,1} + \Delta_{2,2} + \Delta_{2,3} \quad (\text{SA-2})$$

where

$$\begin{aligned} \Delta_{2,1} &= n^{-\mathbf{a}_{\beta,h}} \sum_{i=1}^n g'_i(1, \pi) \left( \frac{M_i}{N_i} - \pi \right), \\ \Delta_{2,2} &= n^{-\mathbf{a}_{\beta,h}} \sum_{i=1}^n \frac{T_i - p_i}{p_i} g'_i(1, \pi) \left( \frac{M_i}{N_i} - \pi \right), \\ \Delta_{2,3} &= n^{-\mathbf{a}_{\beta,h}} \sum_{i=1}^n \frac{T_i Y_i''(1, \eta_i^*)}{2p_i} \left( \frac{M_i}{N_i} - \pi \right)^2 \end{aligned}$$

where  $\eta_i^*$  is some random quantity between  $\frac{M_i}{N_i}$  and  $\pi$ . Define  $b_i = \sum_{j \neq i} \frac{E_{ij}}{N_j} Y'_j(1, \pi)$ . Then by reordering the terms,

$$\Delta_{2,1} = n^{-\mathbf{a}_{\beta,h}} \sum_{i=1}^n b_i (W_i - \pi).$$

**Lemma SA-3.** *Suppose Assumption 1,2,3 hold. Then condition on  $\mathbf{U}$  such that  $A(\mathbf{U}) \in \mathcal{A} = \{A \in \mathbb{R}^{n \times n} : \min_{i \in [n]} \sum_{j \neq i} A_{ij} \geq 32 \log n\}$ ,*

$$\Delta_{2,2} = O_{\psi_2, tc} \left( \log n \max_{i \in [n]} \mathbb{E}[N_i | \mathbf{U}]^{-1/2} \right) + O_{\psi_{\beta, \gamma}, tc}(\sqrt{\log n n^{-\mathbf{r}_{\beta,h}}}).$$

For the term  $\Delta_{2,3}$ , we further decompose it into two parts:

$$\Delta_{2,3} = \Delta_{2,3,1} + \Delta_{2,3,2},$$

where

$$\begin{aligned}\Delta_{2,3,1} &= n^{-\mathbf{a}_{\beta,h}} \sum_{i=1}^n \left[ g_i \left( 1, \frac{M_i}{N_i} \right) - g_i(1, \pi) - g'_i(1, \pi) \left( \frac{M_i}{N_i} - \pi \right) \right], \\ \Delta_{2,3,2} &= n^{-\mathbf{a}_{\beta,h}} \sum_{i=1}^n \frac{1}{2} \frac{W_i - \mathbb{E}[W_i | \mathbf{W}_{-i}]}{p_i} \left[ g_i \left( 1, \frac{M_i}{N_i} \right) - g_i(1, \pi) - g'_i(1, \pi) \left( \frac{M_i}{N_i} - \pi \right) \right].\end{aligned}$$

**Lemma SA-4.** *Suppose 1,2,3 hold. Then condition on  $\mathbf{U}$  such that  $A(\mathbf{U}) \in \mathcal{A} = \{A \in \mathbb{R}^{n \times n} : \min_{i \in [n]} \sum_{j \neq i} A_{ij} \geq 32 \log n\}$ ,*

$$\begin{aligned}\Delta_{2,3,1} - \mathbb{E}[\Delta_{2,3,1} | \mathbf{E}, (f_i)_{i \in [n]}] \\ = O_{\psi_{p_{\beta,h}/2}}(n^{-\mathbf{r}_{\beta,h}}) + O_{\psi_{\beta,h,tc}}(\max_i \mathbb{E}[N_i | \mathbf{U}]^{-1/2}) + O_{\psi_{1,tc}}(n^{-1/2}) \\ + O_{\psi_{2,tc}}(n^{\frac{1}{2} - \mathbf{a}_{\beta,h}} \max_i \mathbb{E}[N_i | \mathbf{U}]^{-1/2}).\end{aligned}$$

**Lemma SA-5.** *Suppose 1,2,3 hold. If  $g_i(1, \cdot)$  and  $g_i(-1, \cdot)$  are 4-times continuously differentiable, then condition on  $\mathbf{U}$  such that  $A(\mathbf{U}) \in \mathcal{A}$ ,*

$$\begin{aligned}\Delta_{2,3,2} - \mathbb{E}[\Delta_{2,3,2} | \mathbf{E}, (f_i)_{i \in [n]}] \\ = O_{\psi_{p_{\beta,h}/2,tc}}((\log n)^{-1/p_{\beta,h}} n^{-2\mathbf{r}_{\beta,h}}) + O_{\psi_{1,tc}}((\log n)^{-1/p_{\beta,h}} (\min_i \mathbb{E}[N_i | \mathbf{U}])^{-1}) \\ + O_{\psi_{1,tc}} \left( n^{1/2 - \mathbf{a}_{\beta,h}} \left( \frac{\max_i \mathbb{E}[N_i | \mathbf{U}]^3}{\min_i \mathbb{E}[N_i | \mathbf{U}]^4} \right)^{1/2} \right) + O_{\psi_{2/(p+1),tc}} \left( n^{\mathbf{r}_{\beta,h}} (\min_i \mathbb{E}[N_i | \mathbf{U}]^{-(p+1)/2}) \right).\end{aligned}$$

## SA-4.2 Hajek Estimator

**Lemma SA-6.** *Suppose Assumption 1, 2 and 3 hold. Then*

$$\hat{\tau}_n - \hat{\tau}_{n,UB} = - \left( \frac{\mathbb{E}[g_i(1, \frac{M_i}{N_i})]}{\pi + 1} + \frac{\mathbb{E}[g_i(-1, \frac{M_i}{N_i})]}{1 - \pi} \right) (1 - \beta(1 - \pi^2))(m - \pi) + O_{\psi_1}(n^{-2\mathbf{r}_{\beta,h}}).$$

## SA-4.3 Stochastic Linearization

**Lemma SA-7.** *Suppose Assumptions 1, 2, and 3 hold. Define*

$$R_i = \frac{g_i(1, \frac{M_i}{N_i})}{1 + \pi} + \frac{g_i(-1, \frac{M_i}{N_i})}{1 - \pi}, \quad Q_i = \mathbb{E} \left[ \frac{G(U_i, U_j)}{\mathbb{E}[G(U_i, U_j) | U_j]} (g'_j(1, \pi) - g'_j(-1, \pi)) | U_i \right].$$

Then,

$$\sup_{t \in \mathbb{R}} |\mathbb{P}(\hat{\tau}_n - \tau_n \leq t) - \mathbb{P}(\frac{1}{n} \sum_{i=1}^n (R_i - \mathbb{E}[R_i] + Q_i)(W_i - \pi) \leq t)| = O \left( \frac{\log n}{\sqrt{n\rho_n}} + \mathbf{r}_{n,\beta} \right),$$

where  $\mathbf{r}_{n,\beta} = \sqrt[4]{n} \sqrt{\log n} (n\rho_n)^{-\frac{p+1}{2}}$  if  $\beta = 1, h = 0$ ; and  $\sqrt{n \log n} (n\rho_n)^{-\frac{p+1}{2}}$  if  $\beta < 1$  or  $h \neq 0$ .

**Lemma SA-8.** *Define Assumptions 1, 2, and 3 hold with  $h = 0, \beta \in [0, 1]$ . Define*

$$R_i = \frac{g_i(1, \frac{M_i}{N_i})}{1 + \pi} + \frac{g_i(-1, \frac{M_i}{N_i})}{1 - \pi}, \quad Q_i = \mathbb{E} \left[ \frac{G(U_i, U_j)}{\mathbb{E}[G(U_i, U_j) | U_j]} (g'_j(1, \pi) - g'_j(-1, \pi)) | U_i \right].$$

Then,

$$\sup_{\beta \in [0,1]} \sup_{t \in \mathbb{R}} |\mathbb{P}(\hat{\tau}_n - \tau_n \leq t) - \mathbb{P}(\frac{1}{n} \sum_{i=1}^n (R_i - \mathbb{E}[R_i] + Q_i)(W_i - \pi) \leq t)| = o(1).$$

## SA-5 Jackknife-Assisted Variance Estimation

**Lemma SA-1.** *Suppose Assumptions 1,2,3,4 hold, and  $n\rho_n^3 \rightarrow \infty$  as  $n \rightarrow \infty$ . Suppose the non-parametric learner  $\widehat{f}$  satisfies  $\widehat{f}(\ell, \cdot) \in C_2([0, 1])$ , and  $|\widehat{f}(\ell, \frac{1}{2}) - f(\ell, \frac{1}{2})| = o_{\mathbb{P}}(1)$ ,  $|\partial_2 \widehat{f}(\ell, \frac{1}{2}) - \partial_2 f(\ell, \frac{1}{2})| = o_{\mathbb{P}}(1)$ , for  $\ell \in \{0, 1\}$ , where the rate in  $o_{\mathbb{P}}(\cdot)$  does not depend on  $\beta$ . Suppose  $\widehat{K}_n$  is the jackknife estimator from Algorithm 2. Then*

$$\widehat{K}_n = \mathbb{E}[(R_i - \mathbb{E}[R_i] + Q_i)^2] + o_{\mathbb{P}}(1),$$

where the rate in  $o_{\mathbb{P}}(1)$  also does not depend on  $\beta$ .

Here we give a local-polynomial based learner  $\widehat{f}$  that satisfies requirements of Lemma SA-1 (hence Theorem 4 in the main paper.)

**Lemma SA-2.** *Use a local polynomial estimator to fit the potential outcome functions: Take*

$$\begin{aligned} \widehat{f}(1, x) &:= \widehat{\gamma}_0 + \widehat{\gamma}_1 x, \\ (\widehat{\gamma}_0, \widehat{\gamma}_1) &:= \arg \min_{\gamma_0, \gamma_1} \sum_{i=1}^n \left( Y_i - \gamma_0 - \gamma_1 \frac{M_i}{N_i} \right)^2 K_h \left( \frac{M_i}{N_i} \right) \mathbb{1}(T_i = 1), \end{aligned}$$

where  $K_h(\cdot) = h^{-1}K(\cdot/h)$  where  $K$  is a kernel function,  $h$  is the optimal bandwidth. Then  $\widehat{f}(1, 0) = f(1, 0) + o_{\mathbb{P}}(1)$ ,  $\partial_2 \widehat{f}(1, 0) = \partial_2 f(1, 0) + o_{\mathbb{P}}(1)$ , the same for control group. Moreover, the rate of convergence can be made not depending on  $\beta$ .

## SA-6 Proof of Main Theorems

### SA-6.1 Proof of Theorem 1

The conclusion follows from the stochastic linearization result in Lemma SA-6, and the Berry-Esseen result for Curie-Weiss magnetization with independent multipliers in Lemma SA-3.

### SA-6.2 Proof of Theorem 2

The conclusion follows from the stochastic linearization result in Lemma SA-6, and the (uniform in  $\beta$ ) Berry-Esseen result for Curie-Weiss magnetization with independent multipliers in Lemma SA-4.

### SA-6.3 Proof of Theorem 3

The uniform approximation for  $\sqrt{n}(\widehat{\beta}_n - 1)$  established in Lemma SA-3 implies

$$\inf_{\beta} \mathbb{P}_{\beta}(\beta \in \mathcal{I}(\alpha_1)) \geq \inf_{\beta} \mathbb{P}_{\beta}(\sqrt{n}(1 - \beta) \geq q) \geq 1 - \alpha_1 + o_{\mathbb{P}}(1).$$

where  $q$  is the  $\alpha_1$  quantile of  $\min\{\max\{\mathbb{T}_{c_{\beta,n,n}}^{-2} - \mathbb{T}_{c_{\beta,n,n}}^2/(3n), 0\}, 1\}$ .

Then by a Bonferroni correction argument, the second step coverage can be lower bounded by

$$\inf_{\beta \in [0, 1]} \mathbb{P}_{\beta}(\tau_n \in \widehat{\mathcal{C}}(\alpha_1, \alpha_2)) \geq \inf_{\beta \in [0, 1]} \mathbb{P}_{\beta}(\tau_n \in \widehat{\mathcal{C}}(\alpha_1, \alpha_2), \beta \in \mathcal{I}(\alpha_1)) - \mathbb{P}_{\beta}(\beta \notin \mathcal{I}(\alpha_1)).$$

Observe that the event  $\tau_n \in \widehat{\mathcal{C}}(\alpha_1, \alpha_2)$  coincides with the event  $\widehat{\tau}_n - \tau_n \in [\inf_{c \in \mathcal{I}(\alpha_1)} \text{law}_{c\beta}(1 - \frac{\alpha_2}{2}; \widehat{s}, n), \sup_{c \in \mathcal{I}(\alpha_1)} \text{law}_{c\beta}(1 - \frac{\alpha_2}{2}; \widehat{s}, n)]$ , where  $\widehat{s} = (\widehat{K}_n, \widehat{K}_n^2)$ . Hence

$$\begin{aligned} & \inf_{\beta \in [0,1]} \mathbb{P}_\beta(\tau_n \in \widehat{\mathcal{C}}(\alpha_1, \alpha_2), \beta \in \mathcal{I}(\alpha_1)) \\ & \geq \inf_{\beta \in [0,1]} \mathbb{P}_\beta(\widehat{\tau}_n - \tau_n \in [\text{law}_{c\beta}(1 - \frac{\alpha_2}{2}; \widehat{s}, n), \text{law}_{c\beta}(1 - \frac{\alpha_2}{2}; \widehat{s}, n)], \beta \in \mathcal{I}(\alpha_1)) \\ & \geq \inf_{\beta \in [0,1]} \mathbb{P}_\beta(\widehat{\tau}_n - \tau_n \in [\text{law}_{c\beta}(1 - \frac{\alpha_2}{2}; \widehat{s}, n), \text{law}_{c\beta}(1 - \frac{\alpha_2}{2}; \widehat{s}, n)]) - \mathbb{P}_\beta(\beta \in \mathcal{I}(\alpha_1)). \end{aligned}$$

Theorem 2 shows that the quantiles of the distributions of  $\widehat{\tau}_n - \tau_n$  can be uniformly approximated by quantiles from  $\text{law}_{c\beta, n}$ , if  $\kappa_1$  and  $\kappa_2$  are correctly specified, and the confidence interval is conservative, if we use upper bounds for  $\kappa_1$  and  $\kappa_2$ . The conclusion then follows.

## SA-6.4 Proof of Theorem 4

The conclusion follows from Theorem 3 and Lemma SA-1.

## SA-7 Proofs

### SA-7.1 Proofs for Section SA-2

#### SA-7.1.1 Proof of Lemma SA-2

Our proof is divided according to the different temperature regimes.

#### The High Temperature Regime.

We introduce the handy notation given by  $F(v) := -\frac{1}{2}v^2 + \log \cosh(\sqrt{\beta}v + h)$ . For the high temperature regime, we note that the term in the exponential can be expanded across its global minimum  $v^*$  (which satisfies the first order stationary point condition given by  $v^* = \sqrt{\beta} \tanh(\sqrt{\beta}v^* + h)$ ) by

$$\begin{aligned} F(v) &= F(v^*) + F'(v^*)(v - v^*) + \frac{1}{2}F^{(2)}(v^*)(v - v^*)^2 + O((v - v^*)^3) \\ &= F(v^*) - \frac{1}{2}(1 - \beta \text{sech}^2(\sqrt{\beta}v^* + h))(v - v^*)^2 + O((v - v^*)^3). \end{aligned}$$

Therefore, to obtain the limit of the expectation, we note that by the Laplace method given similar to the proof of Lemma SA-3 and the definition of  $\mathbf{V}_n := n^{-1/2}\mathbf{U}_n$ :

$$\mathbb{E}[\mathbf{V}_n] = \frac{\int_{\mathbb{R}} v \exp(-nF(v)) dv}{\int_{\mathbb{R}} \exp(-nF(v)) dv} = v^*(1 + O(n^{-1})).$$

Then, we note that for  $\ell \in \mathbb{N}$ , when  $h = 0$  and  $\beta < 1$  we use the Laplace method again to obtain that for all  $\ell \in \mathbb{N}$ ,

$$\begin{aligned} \mathbb{E} \left[ (\mathbf{V}_n - \mathbb{E}[\mathbf{V}_n])^{2\ell} \right] &= \frac{\int_{\mathbb{R}} (v - v^*)^{2\ell} \exp(-n(F(v) - F(v^*))) dv}{\int_{\mathbb{R}} \exp(-n(F(v) - F(v^*))) dv} (1 + O(n^{-1})) \\ &= \frac{1}{\sqrt{\pi}} \left( \frac{2}{n(1 - \beta \text{sech}^2(\sqrt{\beta}v^* + h))} \right)^\ell \Gamma\left(\frac{2\ell + 1}{2}\right) (1 + O(n^{-1})). \end{aligned}$$

Then we can obtain that for all  $t \in \mathbb{R}$ , we have

$$\begin{aligned} \mathbb{E}[\exp(t(V_n - \mathbb{E}[V_n]))] &= \sum_{\ell=0}^{\infty} \frac{t^\ell}{\ell!} \mathbb{E}[(V_n - \mathbb{E}[V_n])^\ell] = \sum_{\ell=0}^{\infty} \frac{t^{2\ell}}{(2\ell)!} \mathbb{E}[(V_n - \mathbb{E}[V_n])^{2\ell}] \\ &\leq \exp\left(\frac{(1+o(1))t^2}{2n(1-\beta \operatorname{sech}^2(\sqrt{\beta}v^*+h))}\right), \end{aligned}$$

which alternatively implies that

$$\|U_n - \mathbb{E}[U_n]\|_{\psi_2} = n^{1/2} \|V_n - \mathbb{E}[V_n]\|_{\psi_2} \leq (1+o(1))(1-\beta \operatorname{sech}^2(\sqrt{\beta}v^*+h))^{\frac{1}{2}}. \quad (\text{SA-3})$$

### The Critical Temperature Regime.

Then we study the critical temperature regime with  $\beta = 1$ . Note that one has  $\mathbb{E}[U_n] = 0$  and for all  $\ell \in \mathbb{N}$  we have

$$\begin{aligned} F(v) &= F(0) + F'(0)v + \frac{1}{2}F^{(2)}(0)v^2 + \frac{1}{6}F^{(3)}(0)v^3 + \frac{1}{24}F^{(4)}(0)v^4 + O(v^5) \\ &= F(0) + \frac{1}{12}v^4 + O(v^5). \end{aligned}$$

Then we can obtain that  $\ell \in \mathbb{N}$ ,

$$\begin{aligned} \mathbb{E}[V_n^{2\ell}] &= \frac{\int_{\mathbb{R}} v^{2\ell} \exp(-nF(v)) dv}{\int_{\mathbb{R}} \exp(-nF(v)) dv} = (1+o(1)) \cdot 2^{\ell-\frac{1}{2}} \cdot 3^{\frac{\ell}{2}+\frac{1}{4}} \frac{\Gamma(\frac{\ell}{2}+\frac{1}{4})}{\Gamma(1/4)} \\ &\leq (1+o(1)) \frac{1}{\sqrt{\pi}} \left(\frac{2^{3/2} \cdot 3^{3/4} \Gamma(3/4)}{n^{1/2} \Gamma(1/4)}\right)^\ell \Gamma\left(\frac{2\ell+1}{2}\right). \end{aligned}$$

And we immediately obtain that

$$\begin{aligned} \mathbb{E}[\exp(tV_n)] &= \sum_{\ell=0}^{\infty} \frac{t^\ell \mathbb{E}[V_n^{2\ell}]}{\Gamma(1+\ell)} \leq \sum_{\ell=0}^{\infty} \frac{1+o(1)}{\Gamma(1+2\ell)} \frac{1}{\sqrt{\pi}} \left(\frac{2^{1/2} \cdot 3^{3/4} \sqrt{2} \Gamma(3/4)}{n^{1/2} \Gamma(1/4)}\right)^\ell \Gamma\left(\frac{2\ell+1}{2}\right) t^\ell \\ &\leq \exp\left(\frac{1+o(1)}{2} t^2 \left(\frac{2^{3/2} \cdot 3^{3/4} \Gamma(3/4)}{n^{1/2} \Gamma(1/4)}\right)\right), \end{aligned}$$

which finally leads to

$$\|V_n\|_{\psi_2} \leq (1+o(1)) \sqrt{\frac{2^{1/2} \cdot 3^{3/4} \Gamma(3/4)}{n^{1/2} \Gamma(1/4)}}. \quad (\text{SA-4})$$

### The Low Temperature Regime.

We shall note that at the low temperature regime the function  $F(v)$  has two symmetric global minima  $v_1 > 0 > v_2$ , satisfying

$$F'(v_1) = F'(v_2) = 0 \quad \Rightarrow \quad v_\ell = \sqrt{\beta} \tanh(\sqrt{\beta}v_\ell + h) \quad \text{for } \ell \in \{1, 2\}.$$

Then we can check that by the Laplace method, for all  $t > 0$  (following the path given by the high temperature regime) we have

$$\begin{aligned} \mathbb{E}[\exp(t(V_n - \mathbb{E}[V_n|V_n > 0]))|V_n > 0] &= \frac{\int_{[0, \infty)} \exp(t(v - v_1) - nF(v)) dv}{\int_{[0, \infty)} \exp(-nF(v)) dv} \\ &= \exp\left(\frac{(1+o(1))t^2}{2n(1-\sqrt{\beta} \operatorname{sech}^2(\sqrt{\beta}v_1))}\right). \end{aligned}$$

Then we similarly obtain that  $\mathbb{E}[\exp(t(\mathbf{V}_n - \mathbb{E}[\mathbf{V}_n|\mathbf{V}_n < 0]))|\mathbf{V}_n < 0] = \exp\left(\frac{(1+o(1))t^2}{2n(1-\sqrt{\beta}\operatorname{sech}^2(\sqrt{\beta}v_1))}\right)$ . Hence we obtain that

$$\begin{aligned}\|\mathbf{V}_n - \mathbb{E}[\mathbf{V}_n|\mathbf{V}_n < 0]|\mathbf{V}_n < 0\|_{\psi_2} &= \|\mathbf{V}_n - \mathbb{E}[\mathbf{V}_n|\mathbf{V}_n > 0]|\mathbf{V}_n > 0\|_{\psi_2} \\ &\leq (1 + o(1))(1 - \beta \operatorname{sech}^2(\sqrt{\beta}v_1))^{\frac{1}{2}}.\end{aligned}\tag{SA-5}$$

### The Drifting Sequence Case.

Then we consider the drifting case.

First consider  $\beta = 1 - cn^{-\frac{1}{2}}$  with  $c \in \mathbb{R}^+$  and  $\beta \geq 0$ . We will show that for any fixed  $n$ ,  $\|W_n\|_{\psi_2}$  is increasing in  $\beta$  when  $\beta \in [0, 1]$ . This will imply that in the drifting case,  $\|W_n\|_{\psi_2}$  will be no larger than its value at the critical regime.

For a comparison argument, denote  $F_\beta(v) = -\frac{1}{2}v^2 + \log \cosh(\sqrt{\beta}v)$ . Let  $0 < \beta_1 < \beta_2 \leq 1$ . Then

$$\frac{\exp(nF_{\beta_2}(v))}{\exp(nF_{\beta_1}(v))} = \exp(n \log \cosh(\sqrt{\beta_2}v) - n \log \cosh(\sqrt{\beta_1}v)),$$

where

$$\frac{d \cosh(\sqrt{\beta_2}v)}{dv \cosh(\sqrt{\beta_1}v)} = \frac{(\sqrt{\beta_2} - \sqrt{\beta_1}) \sinh((\sqrt{\beta_2} - \sqrt{\beta_1})v)}{\cosh^2(\sqrt{\beta_1}v)} > 0.$$

Hence for any  $n \in \mathbb{N}$  and  $t > 0$ ,

$$\mathbb{P}_\beta(|W_n| \geq t) = 2 \frac{\int_t^\infty \exp(nF_\beta(v)) dv}{\int_0^\infty \exp(nF_\beta(v)) dv}$$

increases as  $\beta \in [0, 1]$  increases. This shows that  $\|W_n\|_{\psi_2}$  increases as  $\beta \in [0, 1]$  increases. Together with Equation (SA-4), we have under  $\beta_n = 1 - \frac{c}{\sqrt{n}}$ ,  $0 \leq c \leq \sqrt{n}$ ,

$$\|\mathbf{V}_n\|_{\psi_2} \leq (1 + o(1)) \sqrt{\frac{2^{1/2} \cdot 3^{3/4} \Gamma(3/4)}{n^{1/2} \Gamma(1/4)}},$$

where  $o(\cdot)$  is by an absolute constant.

Then we consider  $\beta = 1 + cn^{-\frac{1}{2}}$ . We shall note that under this situation it is not hard to check that

$$\begin{aligned}\mathbb{E}[\exp(t\mathbf{V}_n)] &= \frac{1}{2} (\mathbb{E}[\exp(t\mathbf{V}_n)|\mathbf{V}_n > 0] + \mathbb{E}[\exp(t\mathbf{V}_n)|\mathbf{V}_n < 0]) \\ &= \frac{1}{2} (\mathbb{E}[\exp(t(\mathbf{V}_n - v_+))|\mathbf{V}_n > 0] \exp(tv_+) + \mathbb{E}[\exp(t(\mathbf{V}_n - v_-))|\mathbf{V}_n < 0] \exp(tv_-)).\end{aligned}$$

Then, under this case we have by Taylor expanding  $F$  at 0 and the fact that  $\sup_{v \in \mathbb{R}} |F^{(5)}(v)| < \infty$ ,

$$f_{\mathbf{V}_n}(v) \propto \sum_{l \in \{-, +\}} \mathbb{1}(v \in \mathcal{C}_l) \exp\left(-cn^{\frac{1}{2}}(v - v_l)^2 - \frac{\sqrt{3c}}{3}n^{\frac{3}{4}}(v - v_l)^3 - \frac{1}{12}n(v - v_l)^4 - O(n(v - v_l)^5)\right).$$

Before we start to upper bound the moments, we first use the fact that  $v_+ = O(n^{-1/4})$  to obtain that

$$\int_{(-v_+, 0)} v^{2\ell} \exp\left(-\sqrt{3c}v^3\right) dv \leq n^{-\frac{1}{4}}v_+^{2\ell} \exp(-\sqrt{3c}n^{-1/4}) = O\left(n^{-1/4-\ell/2}\right).$$

Then we obtain that

$$\begin{aligned}
\mathbb{E}[(V_n - v_+)^{2\ell} | V_n > 0] &= n^{-\frac{\ell}{2}} \frac{\int_{(-v_+, +\infty)} v^{2\ell} \exp\left(-cv^2 - \frac{\sqrt{3c}}{3}v^3 - \frac{1}{12}v^4\right) dv}{\int_{(-v_+, +\infty)} \exp\left(-cv^2 - \frac{\sqrt{3c}}{3}v^3 - \frac{1}{12}v^4\right) dv} (1 + o(1)) \\
&\leq n^{-\frac{\ell}{2}} (1 + o(1)) \frac{\int_{\mathbb{R}} v^{2\ell} \exp(-3cv^2) dv + \int_{(-v_+, +\infty)} v^{2\ell} \exp(-\sqrt{3c}v^3) dv + \int_{\mathbb{R}} v^{2\ell} \exp(-\frac{1}{4}v^4) dv}{\int_{(-v_+, +\infty)} \exp\left(-cv^2 - \frac{\sqrt{3c}}{3}v^3 - \frac{1}{12}v^4\right) dv} \\
&= n^{-\frac{\ell}{2}} (1 + o(1)) \frac{\int_{\mathbb{R}} v^{2\ell} \exp(-3cv^2) dv + \int_{\mathbb{R}^+} v^{2\ell} \exp(-\sqrt{3c}v^3) dv + \int_{\mathbb{R}} v^{2\ell} \exp(-\frac{1}{4}v^4) dv}{\int_{(-v_+, +\infty)} \exp\left(-cv^2 - \frac{\sqrt{3c}}{3}v^3 - \frac{1}{12}v^4\right) dv} + O(n^{-1/4-\ell/2}) \\
&= n^{-\frac{\ell}{2}} (1 + o(1)) \left( C_3 \left(\frac{1}{3c}\right)^\ell \Gamma\left(\ell + \frac{1}{2}\right) + C_4 (3c)^{-\frac{\ell}{3}} \Gamma\left(\frac{2\ell}{3} + \frac{1}{3}\right) + C_5 2^\ell \Gamma\left(\frac{\ell}{2} + \frac{1}{4}\right) \right),
\end{aligned}$$

with  $C_3 := \frac{(3c)^{-1/2}}{3 \int_{(-v_+, +\infty)} \exp\left(-cv^2 - \frac{\sqrt{3c}}{3}v^3 - \frac{1}{12}v^4\right) dv}$ ,  $C_4 = \frac{1}{9 \int_{(-v_+, +\infty)} \exp\left(-cv^2 - \frac{\sqrt{3c}}{3}v^3 - \frac{1}{12}v^4\right) dv}$ ,

and  $C_5 = \frac{2^{-3/2}}{\int_{(-v_+, +\infty)} \exp\left(-cv^2 - \frac{\sqrt{3c}}{3}v^3 - \frac{1}{12}v^4\right) dv}$ . Therefore, we can simply use the definition of the m.g.f. to obtain that

$$\begin{aligned}
\mathbb{E}[\exp(t^2(V_n - v_+)^2) | V_n > 0] &= \sum_{\ell=0}^{\infty} \frac{t^{2\ell} \mathbb{E}[(V_n - v_+)^{2\ell} | V_n > 0]}{\Gamma(2\ell + 1)} \\
&\leq \sum_{\ell=0}^{\infty} \frac{(1 + o(1)) n^{-\ell/2} t^{2\ell}}{\Gamma(2\ell + 1)} \left( C_3 \left(\frac{1}{3c}\right)^\ell \Gamma\left(\ell + \frac{1}{2}\right) + C_4 (3c)^{-\frac{\ell}{3}} \Gamma\left(\frac{2\ell}{3} + \frac{1}{3}\right) + C_5 2^\ell \Gamma\left(\frac{\ell}{2} + \frac{1}{4}\right) \right) \\
&\leq \sum_{\ell=0}^{\infty} \frac{(1 + o(1)) n^{-\ell/2} t^{2\ell}}{\Gamma(2\ell + 1)} \left( C_3 (3c)^{-1} \Gamma\left(\frac{3}{2}\right) + C_4 (3c)^{-1/3} \Gamma(1) + 2C_5 \Gamma\left(\frac{3}{4}\right) \right)^\ell \Gamma\left(\frac{2\ell + 1}{2}\right) \\
&\leq (1 - 2t^2 n^{1/2} / \sigma^2)^{-\frac{1}{2}}, \quad \sigma := \left( C_3 (3c)^{-1} \Gamma\left(\frac{3}{2}\right) + C_4 (3c)^{-1/3} \Gamma(1) + 2C_5 \Gamma\left(\frac{3}{4}\right) \right)^{\frac{1}{2}}.
\end{aligned}$$

Then we use the fact that  $\mathbb{E}[V_n | V_n > 0] = v_+$  to obtain that (here we use proposition 2.5.2 in [7])

$$\mathbb{E}[\exp(t(V_n - v_+)) | V_n > 0] \leq \exp\left(18e^2 n^{-1/2} \sigma^2 t^2\right).$$

Similarly one obtains that  $\mathbb{E}[\exp(t(V_n - v_-)) | V_n < 0] \leq \exp(18e^2 n^{-1/2} \sigma^2 t^2)$ . And hence

$$\mathbb{E}[\exp(tV_n)] \leq \frac{1}{2} (\exp(tv_+) + \exp(-tv_+)) \exp(18e^2 n^{-1/2} \sigma^2 t^2) \leq \exp\left(\frac{1}{2} t^2 v_+^2\right).$$

### SA-7.1.2 Proof for Lemma SA-3 High Temperature

Throughout the proof, we denote by  $\mathbf{C}$  an absolute constant, and  $\mathbf{K}$  a constant that only depends on the distribution of  $X_i$ .

Take  $U_n$  to be a random variable with density

$$f_{U_n}(u) = \frac{\exp\left(-\frac{1}{2}u^2 + n \log \cosh\left(\sqrt{\frac{\beta}{n}}u\right)\right)}{\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}v^2 + n \log \cosh\left(\sqrt{\frac{\beta}{n}}v\right)\right) dv}, \quad u \in \mathbb{R}. \quad (\text{SA-6})$$

By Lemma SA-3, condition on  $\mathbf{U}_n$ ,  $W_i$  are i.i.d Bernouli with

$$\mathbb{P}(W_i = 1|\mathbf{U}_n) = \frac{1}{2}(\tanh(\sqrt{\frac{\beta}{n}}\mathbf{U}_n) + 1).$$

We characterize the conditional mean and variance as

$$\begin{aligned} e(\mathbf{U}_n) &= \mathbb{E}[X_i W_i | \mathbf{U}_n] = \mathbb{E}[X_i] \tanh\left(\sqrt{\frac{\beta}{n}}\mathbf{U}_n\right), \\ v(\mathbf{U}_n) &= \mathbb{V}[X_i W_i | \mathbf{U}_n] = \mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2 \tanh^2\left(\sqrt{\frac{\beta}{n}}\mathbf{U}_n\right). \end{aligned} \quad (\text{SA-7})$$

Moreover, we have  $\mathbb{E}[|X_i^3(W_i - \pi)^3| | \mathbf{U}_n] \leq \mathbb{E}[|X_i|^3]$ .

### Step 1: Conditional Berry-Esseen.

Apply Berry-Esseen Theorem conditional on  $\mathbf{U}_n$ ,

$$\sup_{u \in \mathbb{R}} \sup_{t \in \mathbb{R}} \left| \mathbb{P}(g_n \leq t | \mathbf{U}_n = u) - \Phi\left(\frac{t - \sqrt{n}\mathbb{E}[X_i W_i | \mathbf{U}_n = u]}{\mathbb{V}[X_i W_i | \mathbf{U}_n = u]^{1/2}}\right) \right| \leq \mathbf{c} \frac{\mathbb{E}[|X_i|^3]}{v(\mathbf{U}_n)} n^{-1/2}.$$

Since  $v(\mathbf{U}_n) \geq \mathbb{V}[X_i] + \mathbb{E}[X_i]^2 \text{sech}^2(\sqrt{\beta/n}\mathbf{U}_n)$ , and be Lemma SA-2,  $\|\mathbf{U}_n\|_{\psi_1} \leq \mathbf{C}n^{1/4}$ . Hence

$$\begin{aligned} & d_{\text{KS}}\left(g_n, v(\mathbf{U}_n)^{1/2}\mathbf{Z} + \sqrt{n}e(\mathbf{U}_n)\right) \\ &= \sup_{t \in \mathbb{R}} \left| \int_{-\infty}^{\infty} (\mathbb{P}(g_n \leq t | \mathbf{U}_n = u) - \Phi\left(\frac{t - \sqrt{n}e(\mathbf{U}_n)}{v(\mathbf{U}_n)^{1/2}}\right)) f_{\mathbf{U}_n}(u) du \right| \\ &\leq \mathbf{K}n^{-1/2}. \end{aligned}$$

### Step 2: Approximation for $\mathbf{U}_n$ .

Take  $\mathbf{U} \sim \mathbf{N}(0, (1 - \beta)^{-1})$  independent to  $\mathbf{Z}$ . Consider  $\mathbf{V}_n = n^{-1/2}\mathbf{U}_n$ . Then

$$f_{\mathbf{V}_n}(v) \propto \exp\left(-\frac{1}{2}nv^2 + n \log \cosh\left(\sqrt{\beta}v\right)\right) =: \exp(-n\phi(v)),$$

where  $\phi(v) = -\frac{1}{2}v^2 + \log \cosh(\sqrt{\beta}v)$ . And  $\phi$  is maximized at 0 with  $\phi''(0) = 1 - \beta > 0$ .

We will approximate the integral of  $f_{\mathbf{V}_n}$  by Laplace method. By Equation (5.1.21) in [2],

$$\begin{aligned} \int_{-\infty}^{\infty} \exp(-n\phi(v)) dv &= \sqrt{\frac{2\pi}{n\phi''(0)}} \exp(-n\phi(0)) + O\left(\frac{\exp(-n\phi(0))}{n^{3/2}}\right) \\ &= \sqrt{\frac{2\pi}{n\phi''(0)}} \exp(-n\phi(0)) [1 + O(n^{-1})], \end{aligned}$$

where the  $O(n^{-1})$  term only depends on  $n$  and  $\phi$ . It follows that

$$f_{\mathbf{V}_n}(v) = \sqrt{\frac{n\phi''(0)}{2\pi}} \exp(-n\phi(v) + n\phi(0)) [1 + O(n^{-1})].$$

Then by a change of variable and the fact that  $O(n^{-1})$  term does not depend on  $v$ ,

$$f_{\mathbf{U}_n}(u) = \sqrt{\frac{\phi''(0)}{2\pi}} \exp\left(-n\phi(n^{-1/2}u) + n\phi(0)\right) [1 + O(n^{-1})]. \quad (\text{SA-8})$$

Taylor expanding  $\phi$  at 0, we get

$$-n\phi(n^{-1/2}u) + n\phi(0) = -\frac{\phi''(0)}{2}u^2 - \tanh(\sqrt{\beta}v_* + h) \operatorname{sech}^2(\sqrt{\beta}v_*) \frac{u^3}{3\sqrt{n}} \quad (\text{SA-9})$$

$$= -\frac{1}{2}(1-\beta)(u^2 - \tanh(\sqrt{\beta}v_*) \operatorname{sech}^2(\sqrt{\beta}v_*) \frac{u^3}{3\sqrt{n}}), \quad (\text{SA-10})$$

where  $v_*$  is some quantity between 0 and  $n^{-1/2}u$ . Then

$$\begin{aligned} d_{\text{TV}}(\mathbf{U}_n, \mathbf{U}) &= \int_{-\infty}^{\infty} |f_{\mathbf{U}_n}(u) - f_{\mathbf{U}}(u)| du \\ &\leq \int_{-\infty}^{\infty} \sqrt{\frac{\phi''(0)}{2\pi}} \exp\left(-\frac{1}{2}(1-\beta)u^2\right) \\ &\quad \cdot \left[ \exp\left(-\tanh(\sqrt{\beta}v_*(u)) \operatorname{sech}^2(\sqrt{\beta}v_*(u)) \frac{u^3}{3\sqrt{n}}\right) - 1 \right] du [1 + O(n^{-1})], \end{aligned}$$

where  $v^*(u)$  is some random quantity between 0 and  $n^{-1/2}u$ . We will show that we can restrict the analysis to the region  $[-c_\beta\sqrt{\log n}, c_\beta\sqrt{\log n}]$ , which is where the bulk of mass lies, with  $c_\beta = (1-\beta)^{-1/2}$ . Since  $\mathbf{U} \sim N(0, (1-\beta)^{-1})$ ,  $\mathbb{P}(|\mathbf{U}| \geq c_\beta\sqrt{\log n}) \leq n^{-1}$ . By Lemma SA-2, we also have  $\mathbb{P}(|\mathbf{U}_n| \geq c'_\beta\sqrt{\log n}) \leq n^{-1}$ , where  $c'_\beta$  is a constant that only depends on  $\beta$ . Take  $\mathbf{d}_\beta = \max\{c_\beta, c'_\beta\}$ , and use the boundedness of  $\tanh$  and  $\operatorname{sech}$  and the Lipschitzness of  $\exp$  when restricted to  $[-1, 1]$ , we have

$$\begin{aligned} d_{\text{TV}}(\mathbf{U}_n, \mathbf{U}) &\leq \int_{-\mathbf{d}_\beta\sqrt{\log n}}^{\mathbf{d}_\beta\sqrt{\log n}} \sqrt{\frac{\phi''(0)}{2\pi}} \exp\left(-\frac{1}{2}(1-\beta)u^2\right) \\ &\quad \cdot \left[ \exp\left(-\tanh(\sqrt{\beta}v_*(u)) \operatorname{sech}^2(\sqrt{\beta}v_*(u)) \frac{u^3}{3\sqrt{n}}\right) - 1 \right] du [1 + O(n^{-1})] + O(n^{-1}) \\ &\leq \int_{-\mathbf{d}_\beta\sqrt{\log n}}^{\mathbf{d}_\beta\sqrt{\log n}} \sqrt{\frac{\phi''(0)}{2\pi}} \exp\left(-\frac{1}{2}(1-\beta)u^2\right) c_2 \frac{|u|^3}{\sqrt{n}} du [1 + O(n^{-1})] + O(n^{-1}) \\ &= O(n^{-1/2}). \end{aligned}$$

### Step 3: Data Processing Inequality.

We can use data processing inequality to get

$$d_{\text{KS}}\left(v(\mathbf{U}_n)^{1/2}\mathbf{Z} + \sqrt{n}e(\mathbf{U}_n), v(\mathbf{U})^{1/2}\mathbf{Z} + \sqrt{n}e(\mathbf{U})\right) \leq d_{\text{TV}}(\mathbf{U}_n, \mathbf{U}) = O(n^{-1/2}).$$

### Step 4: Stabilization of Variance.

By independence between  $\mathbf{U}$  and  $Z$ , we have

$$\begin{aligned} & d_{\text{KS}} \left( v(\mathbf{U})^{1/2}Z + \sqrt{ne}(\mathbf{U}), \mathbb{E}[v(\mathbf{U})]^{1/2}Z + \sqrt{ne}(\mathbf{U}) \right) \\ &= \sup_{t \in \mathbb{R}} \mathbb{E} \left[ \Phi \left( \frac{t - \sqrt{ne}(\mathbf{U})}{v(\mathbf{U})^{1/2}} \right) - \Phi \left( \frac{t - \sqrt{ne}(\mathbf{U})}{\mathbb{E}[v(\mathbf{U})]^{1/2}} \right) \right] \\ &\leq \sup_{t \in \mathbb{R}} \mathbb{E} \left[ \left| \phi \left( \frac{t - \sqrt{ne}(\mathbf{U})}{v^*(\mathbf{U})^{1/2}} \right) (t - \sqrt{ne}(\mathbf{U})) (v(\mathbf{U})^{-1/2} - \mathbb{E}[v(\mathbf{U})]^{-1/2}) \right| \right], \end{aligned}$$

where  $v^*(\mathbf{U})$  is some quantity between  $\mathbb{E}[v(\mathbf{U})]$  and  $v(\mathbf{U})$ , and by Equation SA-7,  $v^*(\mathbf{U}) \geq \mathbf{c}^{-1}\mathbb{V}[X_i]$ . It follows from boundedness of  $v(\mathbf{U})$  and Lipschitzness of  $\tanh$  in the expression of  $v(\mathbf{U})$  that

$$\begin{aligned} & d_{\text{KS}} \left( v(\mathbf{U})^{1/2}Z + \sqrt{ne}(\mathbf{U}), \mathbb{E}[v(\mathbf{U})]^{1/2}Z + \sqrt{ne}(\mathbf{U}) \right) \\ &\leq \sup_{t \in \mathbb{R}} \sup_{u \in \mathbb{R}} \left| \phi \left( \frac{t - \sqrt{ne}(u)}{\sqrt{2\mathbb{E}[X_i^2]}} \right) (t - \sqrt{ne}(u)) \right| \frac{1}{2\sqrt{\mathbf{c}^{-1}\mathbb{V}[X_i]}} \mathbb{E} [|v(\mathbf{U}) - \mathbb{E}[v(\mathbf{U})]|] \\ &= O(n^{-1/2}). \end{aligned}$$

### Step 5: Gaussian Approximation for $\sqrt{ne}(\mathbf{U})$ .

In this step, we will show that  $\sqrt{ne}(\mathbf{U})$  can be well-approximated by  $\sqrt{\beta}\mathbf{U}$  and hence  $\sqrt{n}g_n$  can be well-approximated by a Gaussian.

$$\begin{aligned} & d_{\text{KS}} \left( \mathbb{E}[v(\mathbf{U})]^{1/2}Z + \sqrt{ne}(\mathbf{U}), \mathbb{E}[v(\mathbf{U})]^{1/2}Z + \sqrt{\beta}\mathbf{U} \right) \\ &\leq \sup_{t \in \mathbb{R}} \mathbb{E} \left[ \Phi \left( \frac{t - \sqrt{ne}(\mathbf{U})}{\mathbb{E}[v(\mathbf{U})]^{1/2}} \right) - \Phi \left( \frac{t - \sqrt{\beta}\mathbf{U}}{\mathbb{E}[v(\mathbf{U})]^{1/2}} \right) \right] \\ &\leq \frac{\|\phi\|_\infty}{\mathbb{E}[v(\mathbf{U})^{1/2}]} \mathbb{E} \left[ \left| \sqrt{ne}(\mathbf{U}) - \sqrt{\beta}\mathbf{U} \right| \right]. \end{aligned}$$

Taylor expanding  $\tanh$  at 0,

$$\begin{aligned} \sqrt{ne}(\mathbf{U}) &= \mathbb{E}[X_i] \sqrt{n} \tanh \left( \sqrt{\frac{\beta}{n}} \mathbf{U} \right) \\ &= \mathbb{E}[X_i] \sqrt{\beta} \mathbf{U} + O \left( \frac{\beta}{\sqrt{n}} \mathbf{U}^2 \right) + O(n^{-1/2}) \\ &= \mathbb{E}[X_i] \sqrt{\beta} \mathbf{U} + O \left( \frac{\beta}{\sqrt{n}} \mathbf{U}^2 \right) + O(n^{-1/2}), \end{aligned}$$

It follows that  $\mathbb{E} \left[ \left| \sqrt{ne}(\mathbf{U}) - \sqrt{\beta}\mathbf{U} \right| \right] = O(n^{-1/2})$  and hence

$$d_{\text{KS}} \left( \mathbb{E}[v(\mathbf{U})]^{1/2}Z + \sqrt{ne}(\mathbf{U}), \mathbb{E}[v(\mathbf{U})]^{1/2}Z + \mathbb{E}[X_i] \sqrt{\beta}\mathbf{U} \right) = O(n^{-1/2}).$$

Recall  $\mathbf{U} \sim N(0, (1 - \beta)^{-1})$ , hence  $\mathbb{E}[X_i] \sqrt{\beta}\mathbf{U} \sim N(0, \mathbb{E}[X_i]^2 \frac{\beta}{1 - \beta})$ . Moreover,

$$\begin{aligned} \mathbb{E}[v(\mathbf{U})] &= \mathbb{E}[\mathbb{E}[X_i^2] \mathbb{E}[W_i^2 | \mathbf{U}]] - \mathbb{E}[\mathbb{E}[X_i]^2 \mathbb{E}[W_i | \mathbf{U}]^2] \\ &= \mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2 \mathbb{E}[W_i | \mathbf{U}]^2 \\ &= \mathbb{E}[X_i^2] + O(n^{-1/2}), \end{aligned}$$

where the last line is because  $\mathbb{E}[W_i|\mathbf{U}] = \tanh(\sqrt{\beta/n}\mathbf{U})$  and  $\mathbf{U}$  is sub-Gaussian. Since  $\mathbf{Z} \perp \mathbf{U}$ ,

$$d_{\text{KS}}\left(\mathbb{E}[v(\mathbf{U})]^{1/2}\mathbf{Z} + \mathbb{E}[X_i]\sqrt{\beta}\mathbf{U}, N(0, \mathbb{E}[X_i^2] + \mathbb{E}[X_i]^2 \frac{\beta}{1-\beta})\right) = O(n^{-1/2}).$$

Combining the previous five steps, we get

$$d_{\text{KS}}\left(\sqrt{n}g_n, N\left(0, \mathbb{E}[X_i^2] + \mathbb{E}[X_i]^2 \frac{\beta}{1-\beta}\right)\right) = O(n^{-1/2}).$$

### SA-7.1.3 Proof for Lemma SA-3 Critical Temperature

Throughout the proof, we denote by  $\mathbf{C}$  an absolute constant, and  $\mathbf{K}$  a constant that only depends on the distribution of  $X_i$ . The proofs for the critical temperature case will have a similar structure as the proof for the high temperature case, based the same  $\mathbf{U}_n$  defined in Equation (SA-6).

#### Step 1: Conditional Berry-Esseen.

The same argument as in the high-temperature case gives

$$d_{\text{KS}}\left(g_n, v(\mathbf{U}_n)^{1/2}\mathbf{Z} + \sqrt{ne}(\mathbf{U}_n)\right) \leq \mathbf{K}n^{-1/2}.$$

#### Step 2: Approximation for $\mathbf{U}_n$ .

Take  $\mathbf{W}$  to be a random variable with density function

$$f_{\mathbf{W}}(z) = \frac{\sqrt{2}}{3^{1/4}\Gamma(\frac{1}{4})} \exp\left(-\frac{1}{12}z^4\right), \quad z \in \mathbb{R},$$

independent to  $\mathbf{Z}$ . Take  $\mathbf{W}_n = n^{-1/4}\mathbf{U}_n$  and  $\mathbf{V}_n = n^{-1/2}\mathbf{U}_n$ . Again  $f_{\mathbf{V}_n}(v) \propto \exp(-n\phi(v))$ , where  $\phi(v) := -\frac{1}{2}v^2 + \log \cosh(v)$ . In particular,  $\phi^{(v)}(0) = 0$  for all  $0 \leq v \leq 3$ , and  $\phi^{(4)}(0) = -2 < 0$ ,  $\phi^{(5)}(0) = 0$ ,  $\phi^{(6)}(0) = 16 > 0$ . Example 5.2.1 in [2] leads to

$$f_{\mathbf{V}_n}(v) = n^{\frac{1}{4}} \frac{\sqrt{2}}{3^{\frac{1}{4}}\Gamma(\frac{1}{4})} \exp(n\phi(v) - n\phi(0))(1 + o(1)),$$

which implies  $f_{\mathbf{W}_n}(w) = f_{\mathbf{W}}(w)(1 + o(1))$ . Results in [2] do not give a rate, however. We will use a more cumbersome approach to obtain a slightly sub-optimal rate.

By a change of variable,  $f_{\mathbf{W}_n}(w) = \frac{h_n(w)}{\int_{-\infty}^{\infty} h_n(u)du}$ , where  $h_n$  can be written as

$$h_n(w) = \exp\left(-\frac{\sqrt{n}}{2}w^2 + n \log \cosh\left(n^{-\frac{1}{4}}w\right)\right) = \exp\left(-\frac{1}{12}w^4 + g(w)n^{-\frac{1}{2}}w^6\right).$$

The last equality follows from Taylor expanding the term in  $\exp(\cdot)$  at  $w = 0$ , and  $g$  is some bounded function.

$$\int_{-10\sqrt{\log n}}^{10\sqrt{\log n}} h_n(w)dw = I_n(1 + O((\log n)^3 n^{-\frac{1}{2}})), \quad I_n := \int_{-10\sqrt{\log n}}^{10\sqrt{\log n}} \exp\left(-\frac{1}{12}w^4\right)dw$$

Moreover,  $\int_{[-10\sqrt{\log n}, 10\sqrt{\log n}]^c} h_n(w)dw = O(n^{-1/2}) = I_n[1 + O(n^{-\frac{1}{2}})]$ . Hence for denominator, we have  $\int_{-\infty}^{\infty} h_n(w)dw = I_n[1 + O((\log n)^3 n^{-\frac{1}{2}})]$ . It follows that

$$\begin{aligned} & d_{\text{TV}}(\mathbf{W}_n, \mathbf{W}) \\ & \lesssim \int_{-10\sqrt{\log n}}^{10\sqrt{\log n}} I_n^{-1} \exp\left(-\frac{1}{12}w^4\right) n^{-\frac{1}{2}} w^6 dw + \int_{-10\sqrt{\log n}}^{10\sqrt{\log n}} I_n^{-1} O((\log n)^3 n^{-\frac{1}{2}}) dw \\ & \quad + P(|\mathbf{W}_n| \geq 10\sqrt{\log n}) + \mathbb{P}(|\mathbf{W}| \geq 10\sqrt{\log n}) \\ & = O((\log n)^3 n^{-\frac{1}{2}}). \end{aligned}$$

### Step 3: Data Processing Inequality.

We can use data processing inequality to get

$$d_{\text{KS}}\left(v(\mathbf{U}_n)^{1/2} \mathbf{Z} + \sqrt{n}e(\mathbf{U}_n), v(n^{1/4}\mathbf{W})^{1/2} \mathbf{Z} + \sqrt{n}e(n^{1/4}\mathbf{W})\right) \leq d_{\text{TV}}(\mathbf{W}_n, \mathbf{W}) = O(n^{-1/2}).$$

### Step 4: Non-Gaussian Approximation for $n^{\frac{1}{4}}e(n^{\frac{1}{4}}\mathbf{W})$

$$n^{1/4}e(n^{1/4}\mathbf{W}) = \mathbb{E}[X_i] n^{\frac{1}{4}} \tanh\left(n^{-\frac{1}{4}}\mathbf{W}\right) = \mathbb{E}[X_i] \left[\mathbf{W} - O\left(\frac{\mathbf{W}^2}{3\sqrt{n}}\right)\right],$$

where we have use the fact that  $\tanh^{(2)}(0) = 0$ . Hence there exists  $C > 0$  such that for  $n$  large enough, for any  $t > 0$ ,

$$\mathbb{P}\left(\mathbb{E}[X_i] \left[\mathbf{W} + C\frac{\mathbf{W}^2}{\sqrt{n}}\right] \leq t\right) \leq \mathbb{P}\left(n^{1/4}e(n^{1/4}\mathbf{W}) \leq t\right) \leq \mathbb{P}\left(\mathbb{E}[X_i] \left[\mathbf{W} - C\frac{\mathbf{W}^2}{\sqrt{n}}\right] \leq t\right). \quad (0)$$

We have showed that there exists  $c > 0$  such that

$$\mathbb{P}(|\mathbf{W}| \geq c\sqrt{\log n}) \leq n^{-1/2}, \quad (1)$$

in which case  $\mathbf{W}^2/\sqrt{n} \leq 1$  for large enough  $n$ . Hence for large enough  $n$  if  $t/\mathbb{E}[X_i] > c\sqrt{\log n} + 1$ , then

$$\mathbb{P}\left(\mathbf{W} + C\frac{\mathbf{W}^2}{\sqrt{n}} \leq \frac{t}{\mathbb{E}[X_i]}, |\mathbf{W}| \leq c\sqrt{\log n}\right) - \mathbb{P}\left(\mathbf{W} \leq \frac{t}{\mathbb{E}[X_i]}, |\mathbf{W}| \leq c\sqrt{\log n}\right) = 0. \quad (2)$$

If  $0 < t/\mathbb{E}[X_i] < c\sqrt{\log n} + 1$ , then

$$\begin{aligned} & \left| \mathbb{P}\left(\mathbf{W} + \frac{\mathbf{W}^2}{\sqrt{n}} \leq \frac{t}{\mathbb{E}[X_i]}, |\mathbf{W}| \leq c\sqrt{\log n}\right) - \mathbb{P}\left(\mathbf{W} \leq \frac{t}{\mathbb{E}[X_i]}, |\mathbf{W}| \leq c\sqrt{\log n}\right) \right| \\ & \leq \mathbb{P}\left(\frac{t}{\mathbb{E}[X_i]} \leq \mathbf{W} \leq \frac{1 - \sqrt{1 - 4n^{-1/2}t/\mathbb{E}[X_i]}}{2n^{-1/2}}, |\mathbf{W}| \leq c\sqrt{\log n}\right). \end{aligned}$$

Now we study  $g(x; \alpha) = (1 - \sqrt{1 - 4x\alpha})/(2x)$ ,  $x > 0$ . Then  $\sup_{\alpha \leq \frac{1}{4}} \sup_{0 \leq x \leq \frac{1}{2}} |\theta'(x; \alpha)| \leq 2$  and  $g(0; \alpha) = \alpha$ . Since for large enough  $n$ ,  $0 < t/\mathbb{E}[X_i] < c\sqrt{\log n} + 1 \leq \frac{1}{4}$  and  $0 \leq n^{-1/2} \leq \frac{1}{2}$ , we have  $\frac{1 - \sqrt{1 - 4n^{-1/2}t/\mathbb{E}[X_i]}}{2n^{-1/2}} \leq t/\mathbb{E}[X_i] + 2n^{-1/2}$ . Hence if  $0 < t/\mathbb{E}[X_i] < c\sqrt{\log n} + 1$ ,

$$\left| \mathbb{P}\left(\mathbf{W} + \frac{\mathbf{W}^2}{\sqrt{n}} \leq \frac{t}{\mathbb{E}[X_i]}, |\mathbf{W}| \leq c\sqrt{\log n}\right) - \mathbb{P}\left(\mathbf{W} \leq \frac{t}{\mathbb{E}[X_i]}, |\mathbf{W}| \leq c\sqrt{\log n}\right) \right| = O(n^{-1/2}). \quad (3)$$

Combining (1), (2), (3),

$$\sup_{t>0} \left| \mathbb{P} \left( W + \frac{W^2}{\sqrt{n}} \leq \frac{t}{\mathbb{E}[X_i]} \right) - \mathbb{P} \left( W \leq \frac{t}{\mathbb{E}[X_i]} \right) \right| = O(n^{-1/2}).$$

By similar argument, we can show

$$\sup_{t>0} \left| \mathbb{P} \left( W - \frac{W^2}{\sqrt{n}} \leq \frac{t}{\mathbb{E}[X_i]} \right) - \mathbb{P} \left( W \leq \frac{t}{\mathbb{E}[X_i]} \right) \right| = O(n^{-1/2}).$$

Noticing that  $W$  and  $-W$  have the same distribution, the above two inequalities also hold for  $t \leq 0$ . Hence it follows from (0) that

$$d_{\text{KS}} \left( n^{1/4} e(n^{1/4} W), \mathbb{E}[X_i] W \right) = O(n^{-1/2}).$$

**Step 5: Vanishing Variance Term.** Denote by  $f_{W+n^{-1/4}Z}$  the density of  $W + n^{-1/4}Z$ . Then

$$f_{W+n^{-1/4}Z}(y) = \int_{-\infty}^{\infty} \frac{\sqrt{2}}{3^{1/4}\Gamma(\frac{1}{4})} \exp\left(-\frac{1}{12}(y-x)^4\right) \frac{\exp(-\sqrt{n}x^2/2)}{\sqrt{2\pi n^{-1/2}}} dx.$$

We will use Laplace method to show  $f_{W+n^{-1/4}Z}$  is close to  $f_W$ . However, to get uniformity over  $y$ , we need to work harder than in the high temperature case. Define  $\varphi(x) = x^2/2$  and  $g_y(t) = \exp(-(t-y)^4/12)$ . Consider

$$I_{y,+}(\lambda) = \int_0^{\infty} g_y(t) \exp(-\lambda\varphi(t)) dt, \quad I_{y,-}(\lambda) = \int_{-\infty}^0 g_y(t) \exp(-\lambda\varphi(t)) dt.$$

Following Section 5.1 in [2], take  $\tau > 0$  such that  $\varphi(t) = \tau$ , by a change of variable,

$$I_{y,+}(\lambda) = \exp(-\lambda\varphi(0)) \int_0^{\infty} \left[ \frac{g_y(t)}{\varphi'(t)} \Big|_{t=\varphi^{-1}(\tau)} \right] \exp(-\lambda\tau) d\tau = \int_0^{\infty} \frac{\exp(-(\sqrt{2\tau}-y)^4/12)}{\sqrt{2\tau}} \exp(-\lambda\tau) d\tau.$$

To get rate of convergence uniformly in  $y$ , we follow the proof of Watson's Lemma but consider only up to first order term. Taylor expanding  $x \mapsto \exp(-x^4)/12$  up to first order at  $y$ , we have

$$\frac{\exp(-(\sqrt{2\tau}-y)^4/12)}{\sqrt{2\tau}} = \frac{\exp(-y^4/12)}{\sqrt{2\tau}} + \frac{1}{3} \exp(-y^4/12) y^3 + \frac{h_y(\tau^*)}{2} \sqrt{2\tau},$$

where  $\tau^*$  is some quantity between 0 and  $\sqrt{2\tau}$  and

$$h_y(u) = -\exp(-(u-y)^4/12)(u-y)^2 + \frac{1}{9} \exp(-(u-y)^4/12)(u-y)^6.$$

In particular, we have  $\sup_{y \in \mathbb{R}} \sup_{u \in \mathbb{R}} |h_y(u)| < C$  for some absolute constant  $C$ . Then

$$\sup_{y \in \mathbb{R}} \left| \int_0^{\infty} \frac{h_y(\tau^*)}{2} \sqrt{2\tau} \exp(-\lambda\tau) d\tau \right| \leq \frac{C}{\sqrt{2}} \Gamma\left(\frac{3}{2}\right) \lambda^{-3/2}, \quad \forall \lambda > 0.$$

Evaluating the first two terms, we get

$$\sup_{y \in \mathbb{R}} \left| I_{y,+}(\lambda) - \sqrt{\frac{\pi}{2\lambda}} \exp(-y^4/12) - \int_0^{\infty} \frac{1}{3} \exp(-y^4/12) y^3 \exp(-\lambda\tau) d\tau \right| \leq \frac{C}{\sqrt{2}} \Gamma\left(\frac{3}{2}\right) \lambda^{-3/2}, \quad \forall \lambda > 0.$$

Similarly, for  $I_{y,-}$ , change of variable by taking  $\tau < 0$  such that  $\varphi(t) = \tau$ , we have

$$\sup_{y \in \mathbb{R}} \left| I_{y,-}(\lambda) - \sqrt{\frac{\pi}{2\lambda}} \exp(-y^4/12) + \int_0^\infty \frac{1}{3} \exp(-y^4/12) y^3 \exp(-\lambda\tau) d\tau \right| \leq \frac{C}{\sqrt{2}} \Gamma\left(\frac{3}{2}\right) \lambda^{-3/2}, \forall \lambda > 0.$$

Combining the two parts, we get

$$\sup_{y \in \mathbb{R}} \left| \int_{-\infty}^\infty g_y(t) \exp(-\lambda\varphi(t)) dt - \sqrt{\frac{2\pi}{\lambda}} \exp(-y^4/12) \right| \leq C\sqrt{2}\Gamma\left(\frac{3}{2}\right) \lambda^{-3/2}, \quad \forall \lambda > 0.$$

Now take  $\lambda = \sqrt{n}$  and multiply both sides by  $\frac{n^{1/4}}{3^{1/4}\Gamma(\frac{1}{4})\sqrt{\pi}}$ , we get

$$\sup_{y \in \mathbb{R}} \left| f_{W+n^{-1/4}Z}(y) - \frac{\sqrt{2}}{3^{1/4}\Gamma(\frac{1}{4})} \exp(-y^4/12) \right| \leq C \frac{\sqrt{2}\Gamma(\frac{3}{2})}{3^{1/4}\Gamma(\frac{1}{4})\sqrt{\pi}} n^{-1/2}.$$

By a truncation argument, we have

$$\begin{aligned} d_{\text{KS}}(W + n^{-1/4}Z, W) &\leq d_{\text{TV}}(W + n^{-1/4}Z, W) \\ &= \int_{-\sqrt{\log n}}^{\sqrt{\log n}} |f_{W+n^{-1/4}Z}(y) - f_W(y)| dy + \mathbb{P}(|W + n^{-1/4}Z| \geq \sqrt{\log n}) \\ &\quad + \mathbb{P}(|W| \geq \sqrt{\log n}) \\ &\leq C\sqrt{n^{-1} \log n}. \end{aligned}$$

Together with the fact that

$$\begin{aligned} n^{-1/4}v(n^{1/4}W) &= n^{-1/4}(\mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2 \tanh^2(\sqrt{\beta}n^{-1/4}W))^{1/2} \\ &= n^{-1/4}\mathbb{E}[X_i^2]^{1/2}(1 + O_{\psi_2}(n^{-1/4})), \end{aligned}$$

we know

$$d_{\text{KS}}(n^{-1/4}v(n^{1/4}W)^{1/2}Z + n^{1/4}e(n^{1/4}W), W) = O(\sqrt{\log n} n^{-1/2}).$$

Putting together all previous steps, we have

$$d_{\text{KS}}(n^{1/4}g_n, \mathbb{E}[X_i]W) = O((\log n)^3 n^{-1/2}).$$

#### SA-7.1.4 Proof for Lemma SA-3 Low Temperature

Throughout the proof, we denote by  $C$  an absolute constant, and  $K$  a constant that only depends on the distribution of  $X_i$ . The proofs are based on essentially the same argument as in the high temperature case.

Instead of using sub-Gaussianity of  $U_n$ , here we use  $U_n$  is sub-Gaussian condition on  $U_n \in \mathcal{I}_\ell$ ,  $\ell \in \{-, +\}$ . In particular, the previous step 2 by:

#### Step 2: Approximation for $U_n$ .

In case  $\beta > 1$ ,  $\phi(v) = \frac{1}{2}v^2 - \log(\cosh(\sqrt{\beta}v))$  has two global minimum  $v_+$  and  $v_-$ , which are the two solutions of  $v - \sqrt{\beta} \tanh(\sqrt{\beta}v) = 0$ . We want to show  $\phi^{(2)}(v_+) = \phi^{(2)}(v_-) = 1 - \beta + v_+^2 > 0$ .

It suffices to show  $v_+ > \sqrt{\beta - 1}$ . Since  $\phi'(v) < 0$  for  $v \in (0, v_+)$  and  $\phi'(v) > 0$  for  $v \in (v_+, \infty)$ , it suffices to show  $\phi'(\sqrt{\beta - 1}) < 0$ . But

$$\phi'(\sqrt{\beta - 1}) < 0 \Leftrightarrow \sqrt{\beta - 1} - \sqrt{\beta} \tanh(\sqrt{\beta(\beta - 1)}) < 0 \Leftrightarrow \beta > 1.$$

Hence  $\phi^{(2)}(v_+) = \phi^{(2)}(v_-) > 0$ . Observe that on  $\mathcal{I}_- = (-\infty, 0)$  and  $\mathcal{I}_+ = (0, \infty)$  respectively, the absolute minimum of  $\phi$  occurs at  $v_-$  and  $v_+$ , and  $\phi'$  is non-zero on  $\mathcal{I}_-$  and  $\mathcal{I}_+$  except at  $v_-$  and  $v_+$ . Hence we can apply Laplace method (Equation 5.1.21 in [2]) separately on  $\mathcal{I}_-$  and  $\mathcal{I}_+$  to get

$$\begin{aligned} \int_{-\infty}^0 \exp(-n\phi(v)) dv &= \sqrt{\frac{2\pi}{n\phi^{(2)}(v_-)}} \exp(-n\phi(v_-))(1 + O(n^{-1})), \\ \int_0^{\infty} \exp(-n\phi(v)) dv &= \sqrt{\frac{2\pi}{n\phi^{(2)}(v_+)}} \exp(-n\phi(v_+))(1 + O(n^{-1})). \end{aligned}$$

It follows from the definition of  $f_{\mathbf{U}_n}$  and a change of variable that the density of  $\mathbf{U}_n = \sqrt{n}\mathbf{V}_n$  can be approximated by

$$f_{\mathbf{U}_n}(u) = \sum_{l=+,-} \mathbb{1}(u \in \mathcal{C}_l) \sqrt{\frac{\phi^{(2)}(v_-)}{8\pi}} \exp(-n\phi(n^{-1/2}u) + n\phi(n^{-1/2}u_l))(1 + O(n^{-1})),$$

where  $u_l = \sqrt{n}v_l, l \in \{+, -\}$ . Since  $\mathbb{P}(\mathbf{U}_n \in \mathcal{I}_+) = \mathbb{P}(\mathbf{U}_n \in \mathcal{I}_-) = \frac{1}{2}$ , condition on  $\mathbf{U}_n \in \mathcal{I}_+$ ,

$$f_{\mathbf{U}_n|\mathbf{U}_n \in \mathcal{I}_+}(u) = \sqrt{\frac{\phi^{(2)}(v_+)}{2\pi}} \exp(-n\phi(n^{-1/2}u) + n\phi(n^{-1/2}u_+))(1 + O(n^{-1})).$$

It then follows from Equation SA-9 that if we define  $\mathbf{U}_+$  to be a random variable with density

$$f_{\mathbf{U}_+}(u) = \sqrt{\frac{1 - \beta + v_+^2}{2\pi}} \exp(-(1 - \beta + v_+^2)(u - u_+)^2/2),$$

then by Taylor expanding  $\phi$  at  $v_+ = n^{-1/2}u_+$  and a similar argument as in the proof for high temperature case,

$$d_{\text{TV}}(\mathbf{U}_n|\mathbf{U}_n \in \mathcal{I}_+, \mathbf{U}_+) = O(n^{-1/2}).$$

The rest follows from the same argument as in the proof for high temperature case and is sub-Gaussianity of  $\mathbf{U}_n$  condition on  $\mathbf{U}_n \in \mathcal{I}_\ell, \ell \in \{-, +\}$ .

### SA-7.1.5 Proof for Lemma SA-4 Drifting from High Temperature

Throughout the proof, we denote by  $\mathbf{C}$  an absolute constant, and  $\mathbf{K}$  a constant that only depends on the distribution of  $X_i$ .

Let  $\mathbf{U}_n(c), e(\mathbf{U}_n(c)), v(\mathbf{U}_n(c))$  be the latent variable, conditional mean, and conditional variance as previously defined when  $\beta_n = 1 + cn^{-\frac{1}{2}}, c < 0$ . For notational simplicity, we abbreviate the  $c$ , and call them  $\mathbf{U}_n, e(\mathbf{U}_n), v(\mathbf{U}_n)$  respectively. By Lemma SA-2,  $\|\mathbf{U}_n\|_{\psi_2} \leq \mathbf{C}n^{1/4}$ .

#### Step 1: Conditional Berry-Esseen.

Apply Berry-Esseen Theorem conditional on  $U_n$  in the same way as in the high temperature case, we get

$$d_{\text{KS}}\left(g_n, v(U_n)^{1/2}Z + \sqrt{ne}(U_n)\right) \leq Kn^{-1/2}.$$

**Step 2: Non-Normal Approximation for  $n^{-1/4}U_n$ .**

Consider  $W_n = n^{-1/4}U_n$ . Then  $f_{W_n}(w) = I_n(c)^{-1}h_n(w)$ , with  $I_n(c) = \int_{-\infty}^{\infty} h_n(w)dw$ , and

$$h_n(w) = \exp\left(-\frac{\sqrt{n}}{2}w^2 + n \log \cosh\left(n^{-1/4}\sqrt{\beta_n}w\right)\right) = \exp\left(-\frac{c}{2}w^2 - \frac{\beta_n^2}{12}w^4 + g(w)\beta_n^3n^{-1/2}w^6\right),$$

where by smoothness of  $\log(\cosh(\cdot))$ ,  $\|\theta\|_{\infty} \leq K$ . Then

$$\int_{-c\sqrt{\log n}}^{c\sqrt{\log n}} h_n(w)dw = \int_{-c\sqrt{\log n}}^{c\sqrt{\log n}} \exp\left(-\frac{c}{2}w^2 - \frac{\beta_n^2}{12}w^4\right)dw[1 + O(\mathbf{C}^6(\log n)^3n^{-1/2})] \quad (\text{SA-11})$$

$$= I(c)[1 + O(\mathbf{C}^6(\log n)^3n^{-1/2})]. \quad (\text{SA-12})$$

Moreover, by a change of variable and the fact that  $\beta_n \leq 1$ ,

$$\begin{aligned} I_n(c) &:= \int_{-\infty}^{\infty} h_n(w)dw = n^{-1/4} \int_{-\infty}^{\infty} \exp\left(-n\left(\frac{v^2}{2} - \log \cosh(\sqrt{\beta_n}v)\right)\right)dv \\ &\leq n^{-1/4} \int_{-\infty}^{\infty} \exp\left(-n\left(\frac{v^2}{2} - \log \cosh(\sqrt{v})\right)\right)dv \leq \mathbf{C}. \end{aligned}$$

Since  $\|W_n(c)\|_{\psi_2} \leq \mathbf{C}$ ,  $I_n(c)^{-1} \int_{(-c\sqrt{\log n}, c\sqrt{\log n})^c} h_n(w)dw \leq \mathbf{C}n^{-1/2}$ . It follows that

$$\int_{(-c\sqrt{\log n}, c\sqrt{\log n})^c} h_n(w)dw \leq \mathbf{C}n^{-1/2}. \quad (\text{SA-13})$$

Combining Equation SA-11 and SA-13, we have  $I_n(c) = I(c)[1 + O(\mathbf{C}^6(\log n)^3n^{-1/2})]$ . It follows that

$$\begin{aligned} &d_{\text{TV}}(W_n, W) \\ &\leq \int_{-c\sqrt{\log n}}^{c\sqrt{\log n}} \left| \frac{h_n(w)}{I_n(c)} - \frac{h(w)}{I(c)} \right| dw + \mathbb{P}(|W_n| \geq c\sqrt{\log n}) + \mathbb{P}(|W| \geq c\sqrt{\log n}) \\ &\leq \int_{-c\sqrt{\log n}}^{c\sqrt{\log n}} \left| \frac{h_n(w) - h(w)}{I(c)} \right| + h_n(w) \left| \frac{1}{I(c)} - \frac{1}{I_n(c)} \right| dw + O(n^{-1/2}) \\ &\leq \int_{-c\sqrt{\log n}}^{c\sqrt{\log n}} \exp\left(-\frac{c}{2}w^2 - \frac{\beta_n^2}{12}w^4\right) \frac{w^6}{\sqrt{n}I(c)} dw + \int_{-c\sqrt{\log n}}^{c\sqrt{\log n}} \frac{1}{I(c)} O(\mathbf{C}^6(\log n)^3n^{-1/2}) dw + O(n^{-1/2}) \\ &\leq \mathbf{C}(\log n)^3n^{-1/2}. \end{aligned}$$

**Step 3: A Reduction through TV-distance Inequality.**

Since  $Z \perp\!\!\!\perp (U_n, W_n)$ , we can use data processing inequality to get

$$\begin{aligned} d_{\text{KS}}\left(n^{-1/4}v(U_n)^{1/2}Z + n^{1/4}e(U_n), n^{-1/4}v(n^{1/4}W)^{1/2}Z + n^{1/4}e(n^{1/4}W)\right) &\leq d_{\text{TV}}(W_n, W) \\ &\leq \mathbf{C}(\log n)^3n^{-1/2}. \end{aligned}$$

**Step 4: Non-Gaussian Approximation for  $n^{\frac{1}{4}}e(n^{\frac{1}{4}}W)$ .**

This is essentially the same as the proof for step 4 from the critical temperature case in Lemma SA-3.

$$d_{\text{KS}}\left(n^{1/4}e(n^{1/4}W), \mathbb{E}[X_i|W]\right) \leq K \frac{\log n}{\sqrt{n}}.$$

**Step 5: Stabilization of Variance.**

Using the same argument as Step 4 in the high temperature case for Lemma SA-3, and  $\|W\| \leq K$ ,

$$d_{\text{KS}}\left(n^{-\frac{1}{4}}v(n^{\frac{1}{4}}W)^{\frac{1}{2}}Z + n^{\frac{1}{4}}e(n^{\frac{1}{4}}W), n^{-\frac{1}{4}}\mathbb{E}[X_i^2]^{\frac{1}{2}}Z + \mathbb{E}[X_i|W]\right) \leq K \frac{\log n}{\sqrt{n}}.$$

The conclusion then follows from putting together the previous five steps.

**SA-7.1.6 Proof for Lemma SA-4 Drifting from Low Temperature**

Consider the same  $U_n$  defined in Equation (SA-6). Recall  $\phi(v) = \frac{v^2}{2} - \log \cosh(\sqrt{\beta_n}v)$ ,  $\phi'(v) = v - \sqrt{\beta_n} \tanh(\sqrt{\beta_n}v)$ ,  $\phi^{(2)}(v) = 1 - \beta_n \operatorname{sech}^2(\sqrt{\beta_n}v)$ . And we take  $v_+ > 0$ ,  $v_- < 0$  to be the two solutions of  $v - \sqrt{\beta_n} \tanh(\sqrt{\beta_n}v) = 0$ .

**Step 2': Non-Normal Approximation for  $n^{-\frac{1}{4}}U_n$ .**

Take  $V_n = n^{-1/2}U_n$ . Then  $f_{V_n}(v) \propto \exp(-n\phi(v))$ . Taylor expanding  $\phi'$  at 0, we know there exists some function  $g$  that is uniformly bounded such that  $\phi'(v) = (1 - \beta_n)v + \frac{1}{3}\beta_n^2v^3 + \beta_n^3g(v)v^5$ . Hence

$$v_+ = \sqrt{\frac{3(\beta_n - 1)}{\beta_n^2}} + O(\beta_n - 1) = \sqrt{3cn^{-1/4}} + O(n^{-1/2}).$$

Taylor expand  $\tanh$  and  $\operatorname{sech}$  at 0,

$$\begin{aligned} \phi^{(2)}(v_+) &= 1 - \beta_n + v_+^2 \\ &= -cn^{-1/2} + 3cn^{-1/2}(1 + O(cn^{-1/2}))^{-2} + O((cn^{-1/2})^{5/2}) \\ &= 2cn^{-1/2}(1 + O(cn^{-1/2})), \\ \phi^{(3)}(v_+) &= 2(\beta_n - v_+^2)v_+^2 \\ &= 2\beta_n^{3/2} \operatorname{sech}^2(\sqrt{\beta_n}v_+) \tanh(\sqrt{\beta_n}v_+) \\ &= 2(1 + O(cn^{-1/2}))(1 + O(v_+^2))(\sqrt{\beta_n}v_+ + O(v_+^3)) \\ &= 2\sqrt{3}cn^{-1/4}(1 + O(cn^{-1/2})), \\ \phi^{(4)}(v_+) &= 2(\beta - v_+^2)(\beta - 3v_+^2) \\ &= 2\beta_n^2 \operatorname{sech}^4(\sqrt{\beta_n}v_+) - 4\beta_n^2 \operatorname{sech}^2(\sqrt{\beta_n}v_+) \tanh^2(\sqrt{\beta_n}v_+) \\ &= 2(1 + O(cn^{-1/2})). \end{aligned}$$

Take  $W_n = n^{1/4}V_n = n^{-1/4}U_n$ ,  $\omega_+ = n^{1/4}v_+ = \sqrt{3c} + O(n^{-1/4})$ , and  $\omega_- = n^{1/4}v_-$ . Define

$$\begin{aligned} &h_{c,n}(w) \\ &= -\frac{\sqrt{n}\phi^{(2)}(v_+)}{2}(w - \omega_{\operatorname{sgn}(w)})^2 - \frac{n^{1/4}\phi^{(3)}(v_+)}{6}(w - \omega_{\operatorname{sgn}(w)})^3 - \frac{\phi^{(4)}(v_+)}{24}(w - \omega_{\operatorname{sgn}(w)})^4. \end{aligned}$$

By a change of variable and Taylor expansion, the density for  $W_n$  satisfies

$$f_{W_n}(w) \propto g_{c,\gamma}(w) = \exp\left(h_{c,n}(w) + O(\|\phi^{(6)}\|_\infty/6!) \frac{(w - w_{\text{sgn}(w)})^6}{\sqrt{n}}\right). \quad (\text{SA-14})$$

By Lemma SA-2, for  $\ell \in \{-, +\}$ , condition on  $W_n \in \mathcal{I}_{c,n,\ell}$ ,  $W_n - \omega_\ell$  is sub-Gaussian with  $\psi_2$ -norm bounded by  $\mathfrak{C}$ . Let  $W_{c,n}$  be a random variable with density at  $w$  proportional to  $\exp(h_{c,n}(w))$ . By similar argument as Equations SA-11 and SA-13,

$$d_{\text{KS}}(W_n | W_n \in \mathcal{I}_{c,n,\ell}, W_{c,n} | W_{c,n} \in \mathcal{C}_l) \leq \mathfrak{C}(\log n)^3 n^{-1/2}.$$

The other steps, *conditional Berry-Esseen, reduction through TV-distance inequality, and non-Gaussian approximation for  $n^{\frac{1}{4}}e(n^{\frac{1}{4}}W_{c,n})$*  can be proceeded in the same way as in the proof for Lemma SA-3, with  $W_n - \omega_\ell$  sub-Gaussian condition on  $W_n \in \mathcal{I}_{c,n,\ell}$  with  $\psi_2$ -norm bounded by  $\mathfrak{C}$ , and respectively for  $W_{c,n}$ .

### SA-7.1.7 Proof for Lemma SA-5 Knife-Edge Representation

Again we take  $U_n$  to be the latent variable from Lemma SA-1, and  $W_n = n^{-1/4}U_n$ . From Step 2 in the proof of Lemma SA-4,  $f_{W_n}(w) = I_n(c)^{-1}h_n(w)$ , with  $I_n(c) = \int_{-\infty}^{\infty} h_n(w)dw$ , and

$$h_n(w) = \exp\left(-\frac{\sqrt{n}}{2}w^2 + n \log \cosh(n^{-\frac{1}{4}}\sqrt{\beta_n}w)\right) = \exp\left(-\frac{c_n}{2}w^2 - \frac{\beta_n^2}{12}w^4 + g(w)\beta_n^3 n^{-\frac{1}{2}}w^6\right),$$

where by smoothness of  $\log(\cosh(\cdot))$ ,  $\|\theta\|_\infty \leq K$ .

**Case 1: When  $\sqrt{n}(\beta_n - 1) = o(1)$ .** We can apply Berry-Esseen conditional on  $U_n$  the same way as in the proof of Lemma SA-4, and its Step 2 can also be applied here to show that if we take  $\tilde{W}_c$  to be a random variable with density proportional to  $\exp(-c_n^2/2w^2 - \beta_n^2/12w^4)$ , then  $d_{\text{KS}}(W_n, \tilde{W}_c) = O((\log n)^3 n^{-1/2})$ . Moreover,  $c_n = o(1)$  and  $\beta_n = 1 - o(1)$ . Hence  $d_{\text{KS}}(W_n, W_0) = o(1)$ . The rest of the proof then follows from Step 3 to Step 5 in the proof for the critical regime case in Lemma SA-3.

**Case 2: When  $\sqrt{n}(1 - \beta_n) \gg 1$ .** Again we still have  $\|U_n\|_{\psi_2} = O(n^{1/4})$ . Similarly as in the previous case, the first two steps in the proof of Lemma SA-4 implies  $d_{\text{KS}}(W_n, \tilde{W}_c) = o(1)$ , where the density of  $W_c$  is proportional to  $\exp(-c_n^2/2w^2 - \beta_n^2/12w^4)$ . Since  $c_n \gg 1$ , the first term in the exponent dominates, and we can show  $d_{\text{KS}}(W_n, W_c^\dagger) = o(1)$ , where  $W_c^\dagger$  has density proportional to  $\exp(-c_n^2/2w^2)$ . Again, we can Taylor expand to get  $n^{1/4}e(n^{1/4}W) = \mathbb{E}[X_i]n^{\frac{1}{4}}\tanh\left(n^{-\frac{1}{4}}W\right) = \mathbb{E}[X_i][W - O(\frac{W^2}{3\sqrt{n}})]$ , and show  $d_{\text{KS}}(n^{1/4}e(n^{1/4}W_c^\dagger), \mathbb{E}[X_i]W_c^\dagger) = o(1)$ . Combining with stablization of variance as in the proof of Lemma SA-2 (high temperature case), we can show

$$d_{\text{KS}}(g_n, n^{-1/4}\mathbb{E}[X_i^2]^{1/2}Z + \mathbb{E}[X_i]W_c^\dagger) = o(1).$$

Since  $Z$  and  $W_c^\dagger$  are independent Gaussian random variables, we also have  $d_{\text{KS}}(g_n/\sqrt{\mathbb{V}[g_n]}, Z) = o(1)$ .

**Case 3: When  $\sqrt{n}(\beta_n - 1) \gg 1$ .** By Lemma SA-4 (2),

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}(n^{\frac{1}{4}}g_n \leq t | m \in \mathcal{I}_{c,\ell}) - \mathbb{P}(n^{-\frac{1}{4}}\mathbb{E}[X_i^2]^{1/2}Z + \beta_n^{\frac{1}{2}}\mathbb{E}[X_i]W_{c,n} \leq t | W_{c,n} \in \mathcal{I}_{c,\ell}) \right| = o(1), \quad (\text{SA-15})$$

where  $W_{c,n}$  has density proportional to  $\exp(h_{c,n}(w))$ , with

$$h_{c,n}(w) = -\frac{\sqrt{n}\phi^{(2)}(v_+)}{2}(w - w_{\text{sgn}(w)})^2 - \frac{n^{1/4}\phi^{(3)}(v_+)}{6}(w - w_{\text{sgn}(w)})^3 - \frac{\phi^{(4)}(v_+)}{24}(w - w_{\text{sgn}(w)})^4,$$

and  $\mathcal{I}_{c,n,-} = (-\infty, K_{c,n,-})$  and  $\mathcal{I}_{c,n,+} = (K_{c,n,+}, \infty)$  such that  $\mathbb{E}[W_{c,n} | W_{c,n} \in \mathcal{I}_{c,n,\ell}] = w_{c,n,\ell}$  for  $\ell \in \{-, +\}$ . Now we calculate the order of the coefficients under  $\sqrt{n}(\beta_n - 1) \gg 1$ . First, suppose  $\beta_n = 1 + cn^\gamma$  for some  $\gamma \in (0, \infty)$  and  $c$  not depending on  $n$ . Then  $v_+ = \sqrt{\frac{3(\beta_n - 1)}{\beta_n^2}} + O(\beta_n - 1) = \sqrt{3cn^{-\gamma/2}} + O(n^{-\gamma})$ . Taylor expand  $\tanh$  and  $\text{sech}$  at 0,

$$\begin{aligned} \phi^{(2)}(v_+) &= 1 - \beta_n + v_+^2 = -cn^{-\gamma} + cn^{-\gamma}3(1 + cn^{-\gamma})^{-2} + O((cn^{-\gamma})^{5/2}) \\ &= 2cn^{-\gamma}(1 + O(cn^{-\gamma})), \\ \phi^{(3)}(v_+) &= 2\beta_n^{3/2} \text{sech}^2(\sqrt{\beta_n}v_+) \tanh(\sqrt{\beta_n}v_+) \\ &= 2(1 + O(cn^{-\gamma}))(1 + O(v_+^2))(\sqrt{\beta_n}v_+ + O(v_+^3)) \\ &= 2\sqrt{3cn^{-\gamma/2}}(1 + O(cn^{-\gamma})), \\ \phi^{(4)}(v_+) &= -2\beta_n^4 \text{sech}^4(\sqrt{\beta_n}v) + 4 \text{sech}^2(\sqrt{\beta_n}v) \tanh^2(\sqrt{\beta_n}v) \\ &= -2(1 + O(cn^{-\gamma})). \end{aligned}$$

We see when  $\gamma = 1/2$ , all of  $\sqrt{n}\phi^{(2)}(v_+)$ ,  $n^{1/4}\phi^{(3)}(v_+)$  and  $\phi^{(4)}(v_+)$  are of order 1. And when  $c_n = \sqrt{n}(\beta_n - 1) \gg 1$ , we have  $\sqrt{n}\phi^{(2)}(v_+) \gg n^{1/4}\phi^{(3)}(v_+) \gg \phi^{(4)}(v_+)$ . Since  $w_+ = n^{1/4}v_+ = \sqrt{3c_n} \gg 1$ , and similarly,  $|w_-| \gg 1$ , condition on  $W_{c,n} \in [n]$ ,  $W_{c,n} - \mathbb{E}[W_{c,n} | W_{c,n} \in [n]]$  is  $\mathbf{C}$ -sub-Gaussian,  $\ell \in \{-, +\}$ . By similar concentration arguments as in the proof for Step 2 in Lemma SA-4 (1), we can show the second order term in  $h_{c,n}$  dominates, and for  $\ell \in \{-, +\}$ ,

$$\sup_{t \in \mathbb{R}} |\mathbb{P}(W_{c,n} - \mathbb{E}[W_{c,n} | W_{c,n} \in \mathcal{I}_\ell] \leq t | W_{c,n} \in \mathcal{I}_\ell) - \Phi(\sqrt{n(1 - \beta_n + v_\ell^2)}t)| = o(1).$$

The conclusion then follows from plugging the (conditional) Gaussian approximation for  $W_{c,n}$  back into Equation (SA-15), and the fact that  $\mathbf{Z}$  is independent to  $W_{c,n}$  and also Gaussian.

## SA-7.2 Proof of Section SA-3

### SA-7.2.1 Proof of Lemma SA-1

Our proof is constructive. We show that consistent estimate of  $n\mathbb{V}[\hat{\tau}_n]$  would imply that one can distinguish between two constructed hypotheses easily. Let  $\mathcal{P}_n$  be the class of distributions of random vectors  $(\mathbf{W} = (W_1, \dots, W_n), \mathbf{Y} = (Y_1, \dots, Y_n))$  taking values in  $\mathbb{R}^{2n}$  that satisfies Assumptions 1,2,3. Consider the following two data generating processes:

$$\begin{aligned} \text{DGP}_0 : \quad & \beta = 0, \quad G(\cdot, \cdot) \equiv 1, \quad \rho_n = 1, \quad Y_i(\cdot, \cdot) = f_i(\cdot, \cdot) + \varepsilon_i, \quad f_i(\cdot, \cdot) \equiv 1, \\ \text{DGP}_1 : \quad & \beta = u, \quad G(\cdot, \cdot) \equiv 1, \quad \rho_n = 1, \quad Y_i(\cdot, \cdot) = f_i(\cdot, \cdot) + \varepsilon_i, \quad f_i(\cdot, \cdot) \equiv 1, \end{aligned}$$

where  $0 < u < 1$ , and in both cases  $(\varepsilon_i : 1 \leq i \leq n)$  are i.i.d  $\mathbf{N}(0, 1)$  random variables, independent to  $\mathbf{W}$ . Denote by  $\mathbb{P}_{0,n}$  and  $\mathbb{P}_{1,n}$  the laws of  $(\mathbf{W}, \mathbf{Y})$  under  $\text{DGP}_0$  and  $\text{DGP}_1$ . Then

$$\begin{aligned} d_{\text{KL}}(\mathbb{P}_{0,n}(\mathbf{W}, \mathbf{Y}), \mathbb{P}_{1,n}(\mathbf{W}, \mathbf{Y})) &= d_{\text{KL}}(\mathbb{P}_{0,n}(\mathbf{W}), \mathbb{P}_{1,n}(\mathbf{W})) + d_{\text{KL}}(\mathbb{P}_{0,n}(\mathbf{Y} | \mathbf{W}), \mathbb{P}_{1,n}(\mathbf{Y} | \mathbf{W})) \\ &= d_{\text{KL}}(\mathbb{P}_{0,n}(\mathbf{W}), \mathbb{P}_{1,n}(\mathbf{W})), \end{aligned}$$

the first line uses chain rule of  $d_{\text{KL}}$ , the second line uses

$$d_{\text{KL}}(\mathbb{P}_{0,n}(\mathbf{Y}|\mathbf{W}), \mathbb{P}_{1,n}(\mathbf{Y}|\mathbf{W})) = d_{\text{KL}}(\mathbb{P}_{0,n}(\mathbf{Y}), \mathbb{P}_{1,n}(\mathbf{Y})) = 0.$$

From Theorem 2.3 (and its proof) in [1],

$$M := \lim_{n \rightarrow \infty} d_{\text{KL}}(\mathbb{P}_{0,n}(\mathbf{W}), \mathbb{P}_{1,n}(\mathbf{W})) < \infty.$$

Hence for large enough  $n$ ,

$$\begin{aligned} d_{\text{TV}}(\mathbb{P}_{0,n}(\mathbf{W}, \mathbf{Y}), \mathbb{P}_{1,n}(\mathbf{W}, \mathbf{Y})) &\leq 1 - \frac{1}{2} \exp(-d_{\text{KL}}(\mathbb{P}_{0,n}(\mathbf{W}, \mathbf{Y}), \mathbb{P}_{1,n}(\mathbf{W}, \mathbf{Y}))) \\ &\leq 1 - \frac{1}{2} \exp(-M). \end{aligned}$$

Le Cam's method (Section 15.2.1 in [8]) gives for large enough  $n$ ,

$$\begin{aligned} \inf_{\widehat{\mathbf{V}}} \sup_{\mathbb{P}_n \in \mathcal{P}_n} \mathbb{E}_{\mathbb{P}_n} [n(\widehat{\mathbf{V}}[\widehat{\tau} - \tau] - \mathbb{V}[\widehat{\tau} - \tau])] \\ \geq n |\mathbb{V}_{\mathbb{P}_{n,0}}[\widehat{\tau} - \tau] - \mathbb{V}_{\mathbb{P}_{n,1}}[\widehat{\tau} - \tau]| (1 - d_{\text{TV}}(\mathbb{P}_{0,n}(\mathbf{W}, \mathbf{Y}), \mathbb{P}_{1,n}(\mathbf{W}, \mathbf{Y}))) \\ \geq \varepsilon \exp(-M)/2, \end{aligned}$$

in the last line we used Theorem 2 (1) to get  $n\mathbb{V}_{\mathbb{P}_{n,0}}[\widehat{\tau} - \tau] - n\mathbb{V}_{\mathbb{P}_{n,1}}[\widehat{\tau} - \tau] = \varepsilon(1 + o(1))$ .

### SA-7.2.2 Proof of Lemma SA-2

The following discussions will be organized according to the three different cases: (1) When  $\beta < 1$ . (2) When  $\beta \geq 1$ ,  $m$  concentrates around 0. (3) When  $\beta \geq 1$  and  $m$  concentrates around two symmetric locations  $w_+ > 0$  and  $w_- < 0$  with  $|w_+| = |w_-|$ .

We have required  $\widehat{\beta} \in [0, 1]$ . For analysis, consider an unrestricted pseud-likelihood estimator,

$$\widehat{\beta}_{\text{UR}} = \arg \max_{\beta \in \mathbb{R}} l(\beta; \mathbf{W}),$$

where  $l(\beta; \mathbf{W})$  is the pseudo log-likelihood given by

$$l(\beta; \mathbf{W}) = \sum_{i \in [n]} \log \mathbb{P}_{\beta}(W_i | \mathbf{W}_{-i}) = \sum_{i \in [n]} -\log \left( \frac{1}{2} W_i \tanh(\beta m_i) + \frac{1}{2} \right).$$

We show that  $l(\beta; \mathbf{W})$  is concave.

$$\begin{aligned} \frac{\partial}{\partial \beta} l(\beta; \mathbf{W}) &= -\frac{1}{n} \sum_{i=1}^n \frac{(n^{-1} \sum_{j \neq i} W_j) W_i \operatorname{sech}^2(\beta n^{-1} \sum_{j \neq i} W_j)}{W_i \tanh(\beta n^{-1} \sum_{j \neq i} W_j) + 1} \\ &= -\frac{1}{n} \sum_{i=1}^n \left( \frac{1}{n} \sum_{j \neq i} W_j \right) (W_i - \tanh(\beta n^{-1} \sum_{j \neq i} W_j)), \end{aligned}$$

and

$$l^{(2)}(\beta; \mathbf{W}) = \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{n} \sum_{j \neq i} W_j \right)^2 \operatorname{sech}^2 \left( \frac{\beta}{n} \sum_{j \neq i} W_j \right) > 0.$$

Hence  $l(\cdot; \mathbf{W})$  is concave everywhere in  $\mathbb{R}$ . This shows  $\widehat{\beta} = \min\{\max\{\widehat{\beta}_{\text{UR}}, 0\}, 1\}$ . Now we study limiting distribution of  $\widehat{\beta}_{\text{UR}}$

## 1. High and critical temperature regime.

To obtain a more precise distribution for  $\widehat{\beta}_{\text{UR}}$ , we use Fermat's condition to obtain that

$$\begin{aligned}
0 &= \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{n} \sum_{j \neq i} W_j \right) \left( W_i - \tanh \left( \widehat{\beta}_{\text{UR}} n^{-1} \sum_{j \neq i} W_j \right) \right) \\
&= \frac{1}{n} \sum_{i=1}^n \left( m - \frac{W_i}{n} \right) \left( W_i - \tanh(\widehat{\beta}_{\text{UR}} m) + \operatorname{sech}^2(\widehat{\beta}_{\text{UR}} m) \frac{\widehat{\beta}_{\text{UR}} W_i}{n} + O(n^{-2}) \right) \\
&= \frac{1}{n} \sum_{i=1}^n \left( m - \frac{W_i}{n} \right) \left( \left( 1 + \operatorname{sech}^2(\widehat{\beta}_{\text{UR}} m) \frac{\widehat{\beta}_{\text{UR}}}{n} \right) W_i - \tanh(\widehat{\beta}_{\text{UR}} m) + O(n^{-2}) \right) \\
&= \left( 1 + \frac{\widehat{\beta}_{\text{UR}}}{n} \operatorname{sech}^2(\widehat{\beta}_{\text{UR}} m) \right) \left( m^2 - \frac{1}{n} \right) - \frac{n-1}{n} m \tanh(\widehat{\beta}_{\text{UR}} m) + O(n^{-2}) m,
\end{aligned}$$

here  $O(\cdot)$ 's are all up to an absolute constant. By Lemma SA-4 with  $X_i = 1$ , we can show  $\mathbb{E}[|(nm)^{-1}|] \leq Cn^{-1/2}$ . By Markov inequality,  $(nm)^{-1} = O_{\mathbb{P}}(n^{-1/2})$ . Taylor expanding  $\tanh$ , we have

$$\begin{aligned}
\widehat{\beta}_{\text{UR}} &= \frac{n}{(n-1)m} \tanh^{-1} \left( m - \frac{1}{nm} \right) \\
&= \frac{n}{(n-1)m} \left( m - \frac{1}{nm} + \frac{1}{3} \left( m - \frac{1}{nm} \right)^3 + O \left( \left( m - \frac{1}{nm} \right)^5 \right) \right) \\
&= 1 - \frac{1}{nm^2} + \frac{m^2}{3} + O_{\mathbb{P}}(n^{-1}), \tag{SA-16}
\end{aligned}$$

where in the above equation, both  $O(\cdot)$  and  $O_{\mathbb{P}}(\cdot)$  are up to absolute constants. The rest of the results are given according to the different temperature regimes.

**(1) The High Temperature Regime.** Using Lemma SA-2 with  $X_i = 1$ , our result for the high temperature regime with  $\beta < 1$  implies that  $n^{\frac{1}{2}} m \xrightarrow{d} N(0, \frac{1}{1-\beta}) \Rightarrow (1-\beta)nm^2 \xrightarrow{d} \chi^2(1)$ . Therefore we conclude that  $\frac{1-\beta}{1-\widehat{\beta}_{\text{UR}}} \xrightarrow{d} \chi^2(1)$ . The conclusion then follows from  $\widehat{\beta} = \min\{\max\{\widehat{\beta}_{\text{UR}}, 0\}, 1\}$ .

**(2) The Critical Temperature Regime.** Using Lemma SA-2 with  $X_i = 1$ , we have  $d_{\text{KS}}(n^{\frac{1}{4}} m, W_0) = o(1)$ . This implies  $n^{\frac{1}{2}}(\widehat{\beta}_{\text{UR}} - 1) \xrightarrow{d} \text{Law}(\frac{W_0^2}{3} - \frac{1}{W_0^2})$ . Since  $W_0 = O_{\mathbb{P}}(1)$ ,  $\mathbb{P}(\widehat{\beta}_{\text{UR}} < 0) = o(1)$ . The conclusion then follows from  $\widehat{\beta} = \min\{\max\{\widehat{\beta}_{\text{UR}}, 0\}, 1\}$ .

## 2. The low temperature regime.

When  $m$  concentrates around  $\pi_+$  and  $\pi_-$  we have when  $m > 0$ , use the fact that  $\pi_\ell = \tanh(\beta\pi_\ell)$  for  $\ell \in \{+, -\}$ ,

$$\begin{aligned}
\widehat{\beta}_{\text{UR}} - \beta &= \frac{(1 - O(n^{-1}))(m - \tanh(\beta m))}{m \operatorname{sech}^2(\beta m)} + mO(\delta^2) + O(n^{-1}) \\
&= \frac{(1 - O(n^{-1}))((m - \pi_\ell) - (\tanh(\beta m) - \tanh(\beta\pi_\ell)))}{\pi_\ell (\operatorname{sech}^2(\beta\pi_\ell) - 2(m - \pi_\ell) \tanh(\beta\pi_\ell) \operatorname{sech}^2(\beta\pi_\ell) + O(m - \pi_\ell)^2) \left( 1 + \frac{m - \pi_\ell}{\pi_\ell} \right)} \\
&\quad + mO(\delta^2) + O(n^{-1}) \\
&= (1 - O(n^{-1})) \frac{(1 - \beta \operatorname{sech}^2(\beta\pi_\ell))(m - \pi_\ell)}{\pi_\ell \operatorname{sech}^2(\beta\pi_\ell)} (1 + O(m - \pi_\ell)) + mO(\delta^2) + O(n^{-1}).
\end{aligned}$$

and the similar argument gives

$$m(\widehat{\beta}_{\text{UR}} - \beta^*) = \frac{1 - \beta^* \operatorname{sech}^2(\beta^* \pi_\ell)}{\operatorname{sech}^2(\beta^* \pi_\ell)} (m - \pi_\ell) + O_{\psi_1}(n^{-1}).$$

The conclusion then Lemma SA-3 (3) and the convergence of  $m$  to  $\pi_+$  or  $\pi_-$ .

### SA-7.2.3 Proof of Lemma SA-3

Again we consider the unrestricted PMLE given by

$$\widehat{\beta}_{\text{UR}} = \arg \max_{\beta \in \mathbb{R}} l(\beta; \mathbf{W}),$$

where  $l(\beta; \mathbf{W})$  is the pseudo log-likelihood given by

$$l(\beta; \mathbf{W}) = \sum_{i \in [n]} \log \mathbb{P}_\beta(W_i | \mathbf{W}_{-i}) = \sum_{i \in [n]} -\log \left( \frac{1}{2} W_i \tanh(\beta m_i) + \frac{1}{2} \right).$$

For  $\beta \in [0, 1]$ , that is  $c_\beta = \sqrt{n}(\beta - 1) \leq 0$ , Equation (SA-16) and the approximation of  $m$  by  $n^{-1/2} \mathbf{Z} + n^{-1/4} \mathbf{W}_c$  from Lemma SA-4 gives

$$\sup_{\beta \in [0, 1]} \sup_{t \in \mathbb{R}} |\mathbb{P}(1 - \widehat{\beta} \leq t) - \mathbb{P}(z_{\beta, n}^{-2} - \frac{3}{n} z_{\beta, n}^2 \leq t)| = o(1).$$

The conclusion follows from the fact that  $x \mapsto \max\{\min\{x, 0\}, 1\}$  is 1-Lipschitz.

## SA-7.3 Proofs for Section SA-4

### SA-7.3.1 Preliminary Lemmas

**Lemma SA-1.** *Suppose  $\pi = \mathbb{E}[W_i]$  where  $\mathbf{W} = (W_i)_{1 \leq i \leq n}$  takes value in  $\{-1, 1\}^n$  and*

$$\mathbb{P}(\mathbf{W} = \mathbf{w}) = \frac{1}{Z} \exp \left( \frac{\beta}{n} \sum_{i < j} W_i W_j + h \sum_{i=1}^n W_i \right), \quad \beta = 1, h = 0.$$

*Suppose either  $h \neq 0$  or  $h = 0, 0 \leq \beta \leq 1$  holds. Then  $\pi = \tanh(\beta\pi + h) + O(n^{-1})$ .*

*Proof.* First, if  $h = 0$ , then  $\pi = \tanh(\beta\pi + h) = 0$ . Now, consider  $\pi \neq 0$ . Using concentration of  $m := \frac{1}{n} \sum_{i=1}^n W_i$  towards  $\pi$  from Lemma SA-3,

$$\begin{aligned} \pi &= \mathbb{E}[\mathbb{E}[W_i | W_{-i}]] = \mathbb{E}[\tanh(\beta m_i + h)] \\ &= \mathbb{E}[\tanh(\beta\pi + h) + \operatorname{sech}^2(\beta\pi + h)(m_i - \pi) - \operatorname{sech}^2(\beta m^* + h) \tanh(\beta m^* + h)(m_i - \pi)^2] \\ &= \tanh(\beta\pi + h) + O(n^{-1}). \end{aligned}$$

□

**Lemma SA-2.** *Suppose Assumption 1, and Assumption 2, 3 hold. Then (1)*

$$\max_{i \in [n]} \left| \frac{M_i}{N_i} - \pi \right| = O_{\psi_{\beta, \gamma}}(n^{-\mathbf{r}_{\beta, h}}) + O_{\psi_2}(N_i^{-1/2}).$$

(2) Define  $A(\mathbf{U}) = (G(U_i, U_j))_{1 \leq i, j \leq n}$ . Condition on  $\mathbf{U}$  such that  $A(\mathbf{U}) \in \mathcal{A} = \{A \in \mathbb{R}^{n \times n} : \min_{i \in [n]} \sum_{j \neq i} A_{ij} \geq 32 \log n\}$ , for large enough  $n$ , for each  $i \in [n]$  and  $t > 0$ ,

$$\mathbb{P} \left( \left| \frac{M_i}{N_i} - \pi \right| \geq 4\mathbb{E}[N_i | \mathbf{U}]^{-1/2} t^{1/2} + C_{\beta, h} n^{-\mathbf{r}_{\beta, h}} t^{\mathbf{p}_{\beta, h}} \mid \mathbf{U} \right) \leq 2 \exp(-t) + n^{-98},$$

where  $C_{\beta, h}$  is some constant that only depends on  $\beta, h$ .

(3) When  $h = 0$ , and  $\beta \in [0, 1]$ , then there exists a constant  $\mathbf{K}$  that does not depend on  $\beta$ , such that for large enough  $n$ , for each  $i \in [n]$  and  $t > 0$ ,

$$\mathbb{P} \left( \left| \frac{M_i}{N_i} - \pi \right| \geq 4\mathbb{E}[N_i | \mathbf{U}]^{-1/2} t^{1/2} + \mathbf{K} n^{-\mathbf{r}_{\beta, h}} t \mid \mathbf{U} \right) \leq 2 \exp(-t) + n^{-98}.$$

*Proof.* Take  $\mathbf{U}_n$  to be a random variable with density

$$f_{\mathbf{U}_n}(u) = \frac{\exp \left( -\frac{1}{2} u^2 + n \log \cosh \left( \sqrt{\frac{\beta}{n}} u + h \right) \right)}{\int_{-\infty}^{\infty} \exp \left( -\frac{1}{2} v^2 + n \log \cosh \left( \sqrt{\frac{\beta}{n}} v + h \right) \right) dv}.$$

Condition on  $\mathbf{U}_n$ ,  $W_i$ 's are i.i.d. Decompose by

$$\frac{M_i}{N_i} - \pi = \sum_{j \neq i} \frac{E_{ij}}{N_i} (W_j - \mathbb{E}[W_j | \mathbf{U}_n]) + \mathbb{E}[W_j | \mathbf{U}_n] - \pi.$$

Condition on  $\mathbf{U}_n$ ,  $W_i$ 's are i.i.d. Berry-Esseen theorem condition on  $\mathbf{U}_n$  and  $\mathbf{E}$  gives,

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P} \left( \frac{M_i}{N_i} - \pi \leq t \mid \mathbf{E} \right) - \mathbb{P} \left( \sqrt{\frac{v(\mathbf{U}_n)}{N_i}} Z + e(\mathbf{U}_n) \leq t \mid \mathbf{E} \right) \right| = O(n^{-\frac{1}{2}}), \quad (\text{SA-17})$$

where  $e(\mathbf{U}_n) := \mathbb{E}[W_i | \mathbf{U}_n] - \pi = \tanh(\sqrt{\beta/n} \mathbf{U}_n + h) - \pi$ , and  $v(\mathbf{U}_n) := \mathbb{V}[W_i - \pi | \mathbf{U}_n]$ . By McDiarmid's inequality,

$$\mathbb{P} \left( \left| \sum_{j \neq i} \frac{E_{ij}}{N_i} (W_j - \mathbb{E}[W_j | \mathbf{U}_n]) \right| \geq 2N_i^{-1/2} t \mid \mathbf{E} \right) \leq 2 \exp(-t^2).$$

Plugging into Equation (SA-17), we can show (1) holds.

Next, we want to show condition on  $\mathbf{U}$  such that  $A(\mathbf{U}) \in \mathcal{A}$ ,  $\mathbb{P}(N_i \leq \mathbb{E}[N_i | \mathbf{U}] / 3 | \mathbf{U}) \leq n^{-100}$ .

Notice that for any  $\mathbf{U}$  such that  $\rho_n \min_{i \in [n]} \sum_{j \neq i} A_{ij}(\mathbf{U}) \rightarrow \infty$ , Condition on  $A$  such that  $A \in \mathcal{A}$ ,  $E_{ij} = \rho A_{ij} \nu_{ij}$ ,  $1 \leq i \leq j \leq n$  are i.i.d Bernoulli random variables, and for each  $i, j$ ,  $\sum_{k \neq i, j} A_{ki} \geq 32 \log n - 1 \geq 31 \log n$  for  $n \geq 3$ . By bounded difference inequality, for all  $t > 0$ ,

$$\mathbb{P} \left( \left| \sum_{k \neq i, j} E_{ki} - \sum_{k \neq i, j} \rho_n A_{ki} \right| \geq \rho_n \sqrt{\sum_{k \neq i, j} A_{i, j}^2 t} \right) \leq 2 \exp(-2t^2).$$

Hence condition on  $A$ , with probability at least  $1 - n^{-100}$ ,

$$\begin{aligned}
\sum_{k \neq i, j} E_{ki} &\geq \sum_{k \neq i, j} \rho_n A_{ki} - 8\sqrt{\log n} \rho_n \sqrt{\sum_{k \neq i, j} A_{ij}^2} \geq \rho_n \sum_{k \neq i, j} A_{ki} - 8\sqrt{\log n} \rho_n \sqrt{\sum_{k \neq i, j} A_{ki}} \\
&\geq \rho_n \sqrt{\sum_{k \neq i, j} A_{ki}} \left( \sqrt{\sum_{k \neq i, j} A_{ki}} - 8\sqrt{\log n} \right) \\
&\geq \rho_n \sqrt{\sum_{k \neq i, j} A_{ki}} \left( \sqrt{\sum_{k \neq i, j} A_{ki}} - 8\sqrt{31^{-1} \sum_{k \neq i, j} A_{ij}} \right) \\
&\geq \rho_n \sum_{k \neq i, j} A_{ij} / 3 \geq \frac{31}{3} \log n,
\end{aligned} \tag{SA-18}$$

and since  $\rho_n A_{i,j} = \mathbb{E}[E_{ij} | \mathbf{U}] \in [0, 1]$ ,  $\sum_{k \neq i, j} E_{ki} + 1 \geq \mathbb{E}[N_j | \mathbf{A}] / 3$ . By Equation SA-18, condition on  $\mathbf{U}$  such that  $A(\mathbf{U}) \in \mathcal{A}$ ,  $\mathbb{P}(N_i \leq \mathbb{E}[N_i | \mathbf{U}] / 3 | \mathbf{U}) \leq n^{-100}$ .

Hence we can disintegrate over the distribution of  $\mathbf{E}$  to get

$$\mathbb{P} \left( \left| \sum_{j \neq i} \frac{E_{ij}}{N_i} (W_j - \mathbb{E}[W_j | \mathbf{U}_n]) \right| \geq 4\mathbb{E}[N_i | \mathbf{U}]^{-1/2} t \middle| \mathbf{U} \right) \leq 2 \exp(-t^2) + n^{-100}.$$

By Equation SA-7 and Lemma SA-2, and the Lipschitzness of  $\tanh$  that

$$\mathbb{E}[W_i | \mathbf{U}_n] - \pi = O_{\psi_{\beta, h}}(n^{-\mathbf{r}_{\beta, h}}).$$

Plugging into Equation (SA-17), we can show (2) holds.

Under the setting of (3), the only part that depends on  $\beta$  in our proof is  $\mathbf{U}_n$ . Since we show in Lemma SA-2  $\|\mathbf{U}_n\|_{\psi_1} \leq Kn^{1/4}$  for some absolute constant  $K$ , which is essentially the  $\beta = 1$  rate, the conclusion of (3) then follows.  $\square$

### SA-7.3.2 Proof of Lemma SA-1

Since we use the conditional probability  $p_i$  in the inverse probability weight, we have

$$\begin{aligned}
\mathbb{E}[\widehat{\tau}_{n, \text{UB}} | (f_i)_{i \in [n]}, \mathbf{E}] &= \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[ \frac{T_i Y_i}{p_i} - \frac{(1 - T_i) Y_i}{1 - p_i} \middle| (f_i)_{i \in [n]}, \mathbf{E} \right] \\
&= \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[ \mathbb{E} \left[ \frac{T_i Y_i}{p_i} - \frac{(1 - T_i) Y_i}{1 - p_i} \middle| \mathbf{T}_{-i}, (f_i)_{i \in [n]}, \mathbf{E} \right] \middle| (f_i)_{i \in [n]}, \mathbf{E} \right],
\end{aligned}$$

and the conclusion follows from  $\mathbb{E}[T_i | \mathbf{T}_{-i}, (f_i)_{i \in [n]}, \mathbf{E}] = p_i$ .

### SA-7.3.3 Proof of Lemma SA-2

First consider the treatment part.

$$n^{-\mathbf{a}_{\beta, h}} \sum_{i=1}^n \frac{T_i}{p_i} g_i(1, \pi) = n^{-\mathbf{a}_{\beta, h}} \sum_{i=1}^n g_i(1, \pi) + n^{-\mathbf{a}_{\beta, h}} \sum_{i=1}^n \frac{T_i - p_i}{p_i} g_i(1, \pi).$$

For the second term, taylor expand  $p_i^{-1}, p_i$  as follows:

$$p_i^{-1} = 1 + \exp(-2\beta m_i - 2h) = 1 + \exp\left(-2\beta \frac{n-1}{n}\pi - 2h\right) - \exp\left(-2\beta \frac{n-1}{n}\pi - 2h\right) 2\beta \left(m_i - \frac{n-1}{n}\pi\right) + \frac{1}{2} \exp(-\xi_i^*) 4\beta^2 \left(m_i - \frac{n-1}{n}\pi\right)^2, \quad (\text{SA-19})$$

where  $\xi_i^*$  is some random quantity that lies between  $4\frac{\beta}{n}\sum_{j\neq i} W_j$  and  $4\frac{\beta}{n}\sum_{j\neq i} \pi$ . Taking the parameters  $c_i^+ = g_i(1, \pi)(1 + \exp(-2\beta\pi - 2h))$ ,  $d^+ = \beta(1 - \tanh(\beta\pi + h))\mathbb{E}[g_i(1, \pi)]$ . Then

$$\begin{aligned} & n^{-\mathbf{a}\beta, h} \sum_{i=1}^n \frac{T_i - p_i}{p_i} g_i(1, \pi) \\ \stackrel{(1)}{=} & n^{-\mathbf{a}\beta, h} \sum_{i=1}^n (T_i - p_i) g_i(1, \pi) (1 + \exp(-2\beta\pi - 2h) - \exp(-2\beta\pi - 2h) 2\beta(m_i - \pi)) \\ & + O_{\psi_{\beta, h, tc}}(n^{-\mathbf{r}\beta, h}) \\ \stackrel{(2)}{=} & n^{-\mathbf{a}\beta, h} \sum_{i=1}^n c_i(T_i - p_i) + O_{\psi_{\beta, h, tc}}((\log n)^{1/2} n^{-\mathbf{r}\beta, h}) \\ \stackrel{(3)}{=} & n^{-\mathbf{a}\beta, h} \sum_{i=1}^n c_i^+ \left[ T_i - \frac{1}{1 + \exp(-2\beta\pi - 2h)} - \frac{2\beta \exp(2\beta\pi + 2h)}{(1 + \exp(2\beta\pi + 2h))^2} (m_i - \pi) \right] \\ & + O_{\psi_{\beta, h, tc}}((\log n)^{1/2} n^{-\mathbf{r}\beta, h}) \\ = & n^{-\mathbf{a}\beta, h} \sum_{i=1}^n \frac{c_i^+}{2} (W_i - \tanh(\beta\pi + h)) \\ & - n^{-\mathbf{a}\beta, h} \sum_{i=1}^n \frac{2\beta \exp(2\beta\pi + 2h)}{(1 + \exp(2\beta\pi + 2h))^2} \left( \frac{1}{n} \sum_{j\neq i} c_j^+ \right) (W_i - \pi) + O_{\psi_{\beta, h, tc}}((\log n)^{1/2} n^{-\mathbf{r}\beta, h}) \\ \stackrel{(4)}{=} & n^{-\mathbf{a}\beta, h} \sum_{i=1}^n [g_i(1, \pi) + (c_i^+/2 - d^+)(W_i - \pi)] + O_{\psi_{\beta, h, tc}}((\log n)^{1/2} n^{-\mathbf{r}\beta, h}). \end{aligned}$$

**Proof of (1):** By Lemma SA-3,  $m - \pi = O_{\psi_{\beta, h}}(n^{-\mathbf{r}\beta, h})$ . The claim follows from Equation SA-19 and a union bound argument.

**Proof of (2):**

$$n^{-\mathbf{a}\beta, h} \sum_{i=1}^n (T_i - p_i) g_i(1, \pi) (m_i - \pi) = \frac{1}{2} (m - \pi) n^{-\mathbf{a}\beta, h} \sum_{i=1}^n (W_i - \tanh(\beta m + h)) g_i(1, \pi) + O(n^{-\mathbf{a}\beta, h}).$$

By Lemma SA-3,

$$m - \pi = O_{\psi_{\beta, h, tc}}(n^{-\mathbf{r}\beta, h}).$$

Taylor expand  $\tanh(x)$  at  $x = \beta\pi + h$ , we have

$$\begin{aligned}
& n^{-\mathbf{a}\beta, h} \sum_{i=1}^n g_i(1, \pi)(W_i - \tanh(\beta m + h)) \\
&= n^{-\mathbf{a}\beta, h} \sum_{i=1}^n g_i(1, \pi)(W_i - \tanh(\beta\pi + h) - \beta \operatorname{sech}^2(\beta\pi + h)(m - \pi) + \tanh(\beta\pi + h) \operatorname{sech}^2(\beta\pi + h)(m - \pi)^2 \\
&\quad + O((m - \pi)^3)) \\
&= O_{\psi_{\beta, h, tc}}(1).
\end{aligned}$$

hence

$$n^{-\mathbf{a}\beta, h} \sum_{i=1}^n (T_i - p_i) g_i(1, \pi)(m_i - \pi) = O_{\psi_{\beta, h, tc}}((\log n)^{1/2} n^{-\mathbf{r}\beta, h}).$$

**Proof of (3):** The first line follows from a Taylor expansion of  $p_i = (1 + \exp(2\beta m_i + 2h))^{-1}$  at  $\pi$ , and  $m_i - \pi = O_{\psi_{\beta, h}}(n^{-\mathbf{r}\beta, h})$ , noticing that  $c_i, \|\psi''\|_\infty$  are bounded. The second line follows by reordering the terms.

**Proof of (4):** By Lemma SA-1,  $\tanh(\beta\pi + h) = \pi + O(n^{-1})$ . By boundedness and i.i.d of  $g_i(1, \pi)$ ,  $\frac{1}{n} \sum_{j \neq i} c_j = \bar{c} + O(n^{-1}) = \mathbb{E}[c_i] + O_{\mathbb{P}}(n^{-1/2}) + O(n^{-1})$ . Similarly, for the control part, taking the parameters  $c_i^- = g_i(-1, \pi)(1 + \exp(2\beta\pi + 2h))$ ,  $d^- = \beta(1 - \tanh(-\beta\pi - h))\mathbb{E}[g_i(-1, \pi)]$ .

$$\begin{aligned}
& -n^{-\mathbf{a}\beta, h} \sum_{i=1}^n \frac{1 - T_i}{1 - p_i} g_i(-1, \pi) \\
&= -n^{-\mathbf{a}\beta, h} \sum_{i=1}^n g_i(-1, \pi) + n^{-\mathbf{a}\beta, h} \sum_{i=1}^n (c_i^- / 2 - d^-)(W_i - \pi) + O_{\psi_{\beta, h, tc}}((\log n)^{1/2} n^{-\mathbf{r}\beta, h}).
\end{aligned}$$

Using Lemma SA-1 again, we can show  $(1 + \exp(-2\beta\pi - 2h))/2 = 1/\pi + O(n^{-1})$  and  $(1 + \exp(2\beta\pi + 2h))/2 = 1/(1 - \pi) + O(n^{-1})$ ,  $\tanh(-\beta\pi - h) = -\pi + O(n^{-1})$ . The result then follows from replacing these quantities in  $c_i^+, c_i^-, d^+, d^-$  by corresponding ones using  $\pi$ .

### SA-7.3.4 Proof of Lemma SA-3

We decompose by  $\Delta_{2,2} = \Delta_{2,2,1} + \Delta_{2,2,2}$ , where

$$\begin{aligned}
\Delta_{2,2,1} &= n^{-\mathbf{a}\beta, h} \sum_{i=1}^n \frac{T_i - \mathbb{E}[p_i]}{\mathbb{E}[p_i]} g'_i(1, \pi) \left( \frac{M_i}{N_i} - \pi \right), \\
\Delta_{2,2,2} &= n^{-\mathbf{a}\beta, h} \sum_{i=1}^n T_i (p_i^{-1} - \mathbb{E}[p_i]^{-1}) g'_i(1, \pi) \left( \frac{M_i}{N_i} - \pi \right).
\end{aligned}$$

Notice that the first term is a quadratic form. Define  $\mathbf{H}$  such that  $H_{ij} = \frac{g'_i(1, \pi) E_{ij}}{2\mathbb{E}[p_i] N_i}$ . Then  $\Delta_{2,2,1} = n^{-\mathbf{a}\beta, h} (\mathbf{W} - \pi)^\top \mathbf{H} (\mathbf{W} - \pi)$ . Take  $\mathbf{U}_n$  to be the latent variable from Lemma SA-1. Then we decompose

$$\Delta_{2,2,1} = \Delta_{2,2,1,a} + \Delta_{2,2,1,b} + \Delta_{2,2,1,c} + \Delta_{2,2,1,d},$$

where

$$\begin{aligned}\Delta_{2,2,1,a} &= n^{-\mathbf{a}\beta,h} (\mathbf{W} - \mathbb{E}[\mathbf{W}|\mathbf{U}_n])^T \mathbf{H} (\mathbf{W} - \mathbb{E}[\mathbf{W}|\mathbf{U}_n]), \\ \Delta_{2,2,1,b} &= n^{-\mathbf{a}\beta,h} (\mathbb{E}[\mathbf{W}|\mathbf{U}_n] - \pi)^T \mathbf{H} (\mathbf{W} - \mathbb{E}[\mathbf{W}|\mathbf{U}_n]), \\ \Delta_{2,2,1,c} &= n^{-\mathbf{a}\beta,h} (\mathbf{W} - \mathbb{E}[\mathbf{W}|\mathbf{U}_n])^T \mathbf{H} (\mathbb{E}[\mathbf{W}|\mathbf{U}_n] - \pi), \\ \Delta_{2,2,1,d} &= n^{-\mathbf{a}\beta,h} (\mathbb{E}[\mathbf{W}|\mathbf{U}_n] - \pi)^T \mathbf{H} (\mathbb{E}[\mathbf{W}|\mathbf{U}_n] - \pi).\end{aligned}$$

Since  $\|\mathbf{H}\|_2 \leq \|\mathbf{H}\|_F \leq \frac{B}{2\pi} \sqrt{n} (\min_i N_i)^{-1/2}$ , we can apply Hanson-Wright inequality conditional on  $\mathbf{U}_n, \mathbf{E}$ ,

$$\Delta_{2,2,1,a} = O_{\psi_1} (n^{\frac{1}{2}-\mathbf{a}\beta,h} (\min_i N_i)^{-1/2}).$$

Since  $g'_i(1, \pi)$ 's are independent to  $W_i$ , by Lemma SA-3,

$$n^{-\mathbf{a}\beta,h} \sum_{i=1}^n (W_i - \pi) g'_i(1, \pi) = O_{\psi_{\beta,h,tc}}(1).$$

By Equation SA-7, Lipschitzness of tanh and Lemma SA-2,  $\mathbb{E}[W_i|\mathbf{U}_n] - \pi = O_{\psi_{\beta,h}}(n^{-\mathbf{r}\beta,h})$ , hence

$$\Delta_{2,2,1,b} = (\mathbb{E}[W_i|\mathbf{U}_n] - \pi) n^{-\mathbf{a}\beta,h} \sum_{i=1}^n \frac{T_i - \mathbb{E}[p_i]}{\mathbb{E}[p_i]} g'_i(1, \pi) = O_{\psi_{\beta,h,tc}} \left( (\log n)^{-1/2} n^{-\mathbf{r}\beta,h} \right).$$

Then by concentration of  $\frac{M_i}{N_i}$  from Lemma SA-2, we have

$$\begin{aligned}|\Delta_{2,2,1,c}| &= \left| \frac{\mathbb{E}[W_i|\mathbf{U}_n] - \pi}{2\mathbb{E}[p_i]} n^{-\mathbf{a}\beta,h} \sum_{i=1}^n g'_i(1, \pi) \left( \frac{M_i}{N_i} - \pi \right) \right| \\ &\leq n^{\mathbf{r}\beta,h} \left| \frac{\mathbb{E}[W_i|\mathbf{U}_n] - \pi}{2\mathbb{E}[p_i]} \right| \cdot \max_{i \in [n]} \left| \frac{M_i}{N_i} - \pi \right| \\ &= O_{\psi_{2,tc}} \left( \log n \max_{i \in [n]} \mathbb{E}[N_i|\mathbf{U}]^{-1/2} \right) + O_{\psi_{\beta,\gamma,tc}}(n^{-\mathbf{r}\beta,h}).\end{aligned}$$

The bound for  $\Delta_{2,2,1,d}$  follows from the definition of  $\mathbf{H}$  and  $\mathbf{U}_n$ ,

$$\Delta_{2,2,1,d} = n^{\mathbf{r}\beta,h} \left( \tanh \left( \sqrt{\frac{\beta}{n}} \mathbf{U}_n + h \right) - \mathbb{E} \left[ \tanh \left( \sqrt{\frac{\beta}{n}} \mathbf{U}_n + h \right) \right] \right)^2 = O_{\psi_{\beta,\gamma}}(n^{-\mathbf{r}\beta,h}).$$

### SA-7.3.5 Proof of Lemma SA-4

Take  $\mathbf{U}_n$  to be the latent variable given in Lemma SA-1. We further decompose by

$$\Delta_{2,3,1} = n^{-\mathbf{a}\beta,h} \sum_{i=1}^n \frac{1}{2} g_i^{(2)}(1, \eta_i^*) \left( \sum_{j \neq i} \frac{E_{ij}}{N_i} (W_i - \pi) \right)^2 = \Delta_{2,3,1,a} + \Delta_{2,3,1,b} + \Delta_{2,3,1,c},$$

where  $\eta_i^*$  is some value between  $\pi$  and  $M_i/N_i$ , and

$$\begin{aligned}\Delta_{2,3,1,a} &= n^{-a_{\beta,h}} \sum_{i=1}^n \frac{1}{2} g_i^{(2)}(1, \eta_i^*) \left( \sum_{j \neq i} \frac{E_{ij}}{N_i} (W_j - \mathbb{E}[W_j | \mathbf{U}_n]) \right)^2, \\ \Delta_{2,3,1,b} &= n^{-a_{\beta,h}} \sum_{i=1}^n \frac{1}{2} g_i^{(2)}(1, \eta_i^*) \left( \sum_{j \neq i} \frac{E_{ij}}{N_i} (W_j - \mathbb{E}[W_j | \mathbf{U}_n]) \right) (\mathbb{E}[W_j | \mathbf{U}_n] - \pi), \\ \Delta_{2,3,1,c} &= n^{-a_{\beta,h}} \sum_{i=1}^n \frac{1}{2} g_i^{(2)}(1, \eta_i^*) (\mathbb{E}[W_j | \mathbf{U}_n] - \pi)^2.\end{aligned}$$

**Part I:**  $\Delta_{2,3,1,c}$ .

$\mathbb{E}[W_i | \mathbf{U}_n, \mathbf{U}] = \tanh\left(\sqrt{\frac{\beta}{n}} \mathbf{U}_n + h\right)$ , hence  $\mathbb{E}[W_i | \mathbf{U}_n] - \pi = O_{\psi_{\beta,h}}(n^{-r_{\beta,h}})$  and  $(\mathbb{E}[W_i | \mathbf{U}_n] - \pi)^2 = O_{\psi_{\beta,h/2}}(n^{-2r_{\beta,h}})$ . It then follows from boundness of  $g_i^{(2)}(1, \eta_i^*)$  that

$$\Delta_{2,3,1,c} = O_{\psi_{\beta,h/2}}(n^{-r_{\beta,h}}).$$

**Part II:**  $\Delta_{2,3,1,b}$ .

Condition on  $\mathbf{U}_n$ ,  $W_i$ 's are i.i.d. By Mc-Diarmid inequality conditional on  $\mathbf{U}_n$  for each  $\sum_{j \neq i} \frac{E_{ij}}{N_i} (W_j - \mathbb{E}[W_j | \mathbf{U}_n])$  and using a union bound over  $i \in [n]$ , for all  $i \in [n]$ , for all  $t > 0$ ,

$$\mathbb{P}\left(|\Delta_{2,3,1,b}| \geq 2 \max_i N_i^{-1/2} n^{r_{\beta,h}} |\mathbb{E}[W_j | \mathbf{U}_n] - \pi| \sqrt{t} \mid \mathbf{U}_n, \mathbf{E}\right) \leq 2n \exp(-t).$$

The tails for  $n^{r_{\beta,h}} (\mathbb{E}[W_j | \mathbf{U}_n] - \pi)$  are also controlled,

$$\mathbb{P}\left(n^{r_{\beta,h}} |\mathbb{E}[W_j | \mathbf{U}_n] - \pi| \geq C_{\beta,h} (\log n)^{1/p_{\beta,h}}\right) \leq n^{-1/2}.$$

Integrate over the distribution of  $\mathbf{U}_n$  and using a union bound, for large  $n$ , for all  $t > 0$ ,

$$\mathbb{P}\left(|\Delta_{2,3,1,b}| \geq 2C_{\beta,h} \max_i N_i^{-1/2} t^{1/p_{\beta,h}} \mid \mathbf{E}\right) \leq 2n \exp(-t) + C_{\beta,h} n^{-1/2}.$$

By Equation SA-18, condition on  $\mathbf{U}$  such that  $A(\mathbf{U}) \in \mathcal{A}$ ,  $\mathbb{P}(N_i \leq \mathbb{E}[N_i | \mathbf{U}]/3 | \mathbf{U}) \leq n^{-100}$ . Hence for such  $\mathbf{U}$ ,

$$\mathbb{P}\left(|\Delta_{2,3,1,b}| \geq 4C_{\beta,h} \max_i \mathbb{E}[N_i | \mathbf{U}]^{-1/2} t^{1/p_{\beta,h}} \mid \mathbf{U}\right) \leq 2n \exp(-t) + C_{\beta,h} n^{-1/2}.$$

In other words, conditional on  $\mathbf{U}$  s.t.  $A(\mathbf{U}) \in \mathcal{A}$ ,

$$\Delta_{2,3,1,b} = O_{\psi_{\beta,h,tc}}(\max_i \mathbb{E}[N_i | \mathbf{U}]^{-1/2}).$$

**Part III:**  $\Delta_{2,3,1,a}$ .

For notational simplicity, we will denote

$$\begin{aligned} A_i &= \frac{1}{2} g_i^{(2)}(1, \eta_i^*) \left( \sum_{j \neq i} \frac{E_{ij}}{N_i} (W_j - \mathbb{E}[W_j | \mathbf{U}_n]) \right)^2 \\ &= \frac{1}{2} \theta \left( \frac{M_i}{N_i} \right) \left( \sum_{j \neq i} \frac{E_{ij}}{N_i} (W_j - \mathbb{E}[W_j | \mathbf{U}_n]) \right)^2 =: F(\mathbf{W}, \mathbf{U}_n), \end{aligned}$$

and since we assume  $g_i(\ell, \cdot)$  is  $C^4$  for  $\ell \in \{-1, 1\}$ , we know  $\theta(\ell, \cdot)$  is  $C^2$  for  $\ell \in \{-1, 1\}$ . Then we can decompose  $\Delta_{2,3,1,a} - \mathbb{E}[\Delta_{2,3,1,a} | \mathbf{E}]$  as

$$\Delta_{2,3,1,a} - \mathbb{E}[\Delta_{2,3,1,a} | \mathbf{E}] = n^{-\mathbf{a}_{\beta,h}} \sum_{i=1}^n (A_i - \mathbb{E}[A_i | \mathbf{U}_n, \mathbf{E}]) + n^{-\mathbf{a}_{\beta,h}} \sum_{i=1}^n (\mathbb{E}[A_i | \mathbf{U}_n, \mathbf{E}] - \mathbb{E}[A_i | \mathbf{E}]).$$

where  $F$  is a function that possibly depends on  $\beta(\mathbf{U})$  and  $\mathbf{E}$ .

**First part of  $\Delta_{2,3,1,a}$ :** The first two terms have a quadratic form in  $W_j - \mathbb{E}[W_j | \mathbf{U}_n]$ , except for the term  $\theta(M_i/N_i)$ . We will handle it via a generalized version of Hanson-Wright inequality. Fix  $\mathbf{U}_n$  and  $\mathbf{E}$ , consider

$$H(\mathbf{W}) = n^{-1/2} \sum_{i=1}^n \frac{1}{2} \theta \left( \frac{M_i}{N_i} \right) \left( \sum_{j \neq i} \frac{E_{ij}}{N_i} (W_j - \mathbb{E}[W_j | \mathbf{U}_n]) \right)^2.$$

Denoting by  $D_k H$  the partial derivative of  $H$  w.r.p to  $W_k$  and  $D_{k,l}$  the mixed partials, then

$$\begin{aligned} D_k H(\mathbf{W}) &= n^{-1/2} \sum_{i \neq k}^n \frac{1}{2} \theta' \left( \frac{M_i}{N_i} \right) \frac{E_{ik}}{N_i} \left( \sum_{j \neq i} \frac{E_{ij}}{N_i} (W_j - \mathbb{E}[W_j | \mathbf{U}_n]) \right)^2 \\ &\quad + n^{-1/2} \sum_{i \neq k}^n \theta \left( \frac{M_i}{N_i} \right) 2 \left( \sum_{j \neq i} \frac{E_{ij}}{N_i} (W_j - \mathbb{E}[W_j | \mathbf{U}_n]) \right) \frac{E_{ik}}{N_i}. \end{aligned}$$

Since we have assumed  $f$  is at least 4-times continuously differentiable, we can apply standard concentration inequalities for  $\sum_{j \neq i} \frac{E_{ij}}{N_i} (W_j - \mathbb{E}[W_j | \mathbf{U}_n])$  to get

$$|\mathbb{E}[D_k H(\mathbf{W}) | \mathbf{U}_n, \mathbf{E}]| \lesssim n^{-1/2} \sum_{i=1}^n E_{ik} N_i^{-3/2}.$$

Hence the gradient of  $H$  is bounded by

$$\begin{aligned} \|\mathbb{E}[DH(\mathbf{W}) | \mathbf{U}_n, \mathbf{E}]\|_2^2 &\lesssim \sum_{k=1}^n n^{-1} \left( \sum_{i=1}^n E_{ik} N_i^{-3/2} \right)^2 \\ &\lesssim \sum_{k=1}^n n^{-1} \left( \sum_{i=1}^n E_{ik} N_i^{-3} + \sum_{j_1=1}^n \sum_{j_2 \neq j_1}^n \frac{E_{j_1 k} E_{j_2 k}}{N_{j_1}^{3/2} N_{j_2}^{3/2}} \right) \\ &\lesssim \frac{\max_i N_i^2}{\min_i N_i^3}. \end{aligned}$$

Moreover, the mix partials are

$$\begin{aligned} D_{k,l}H(\mathbf{W}) &= n^{-1/2} \sum_{i \neq k,l}^n \theta'' \left( \frac{M_i}{N_i} \right) \left( \sum_{j \neq i} \frac{E_{ij}}{N_i} (W_j - \mathbb{E}[W_j | \mathbf{U}_n]) \right)^2 \frac{E_{ik}E_{il}}{N_i^2} \\ &\quad + 2n^{-1/2} \sum_{i=1}^n \theta' \left( \frac{M_i}{N_i} \right) 2 \left( \sum_{j \neq i} \frac{E_{ij}}{N_i} (W_j - \mathbb{E}[W_j | \mathbf{U}_n]) \right) \frac{E_{ik}E_{il}}{N_i^2} \\ &\quad + n^{-1/2} \sum_{i=1}^n \theta \left( \frac{M_i}{N_i} \right) \frac{E_{ik}E_{il}}{N_i^2}. \end{aligned}$$

Hence  $\|D_{k,l}H(\mathbf{W})\|_\infty \lesssim n^{-1/2} \sum_{i=1}^n \frac{E_{ik}E_{il}}{N_i^2}$ . Hence

$$\| \|HF\|_F^2 \|_\infty \lesssim \sum_{k=1}^n \sum_{l=1}^n \left( n^{-1/2} \sum_{i=1}^n \frac{E_{ik}E_{il}}{N_i^2} \right)^2 \lesssim n^{-1} \sum_{i_1=1}^n \sum_{l=1}^n \frac{E_{i_1 l}}{N_{i_1}} \sum_{k=1}^n \frac{E_{i_1 k}}{N_{i_1}} \sum_{i_2=1}^n \frac{E_{i_2 k}}{N_{i_2}} \frac{1}{N_{i_2}} \lesssim \frac{\max_i N_i}{\min_i N_i^2}.$$

Moreover, since  $HF$  is symmetric,

$$\| \|HF\|_2 \|_\infty \leq \| \|HF\|_1 \|_\infty \lesssim \max_k \sum_{l=1}^n n^{-1/2} \sum_{i=1}^n \frac{E_{ik}E_{il}}{N_i^2} \lesssim n^{-1/2} \frac{\max_i N_i}{\min_i N_i}.$$

Hence by Theorem 3 from [4], for all  $t > 0$ ,

$$\begin{aligned} \mathbb{P} \left( \left| n^{-1/2} \sum_{i=1}^n (A_i - \mathbb{E}[A_i | \mathbf{U}_n, \mathbf{E}]) \right| \geq t \mid \mathbf{U}_n, \mathbf{E} \right) \\ \leq \exp \left( -c \min \left( \frac{t^2}{\frac{\max_i N_i^2}{\min_i N_i^3} + \frac{\max_i N_i}{\min_i N_i^2}}, \frac{t}{n^{-1/2} \frac{\max_i N_i}{\min_i N_i}} \right) \right). \end{aligned}$$

By Equation SA-18 and a similar argument for upper bound, for each  $i \in [n]$ , conditional on  $\mathbf{U}$  such that  $A(\mathbf{U}) \in \mathcal{A}$ , with probability at least  $1 - n^{-100}$ ,  $\mathbb{E}[N_i | \mathbf{U}] / 2 \leq N_i \leq 2\mathbb{E}[N_i]$ . Hence for each  $t > 0$ ,

$$\mathbb{P} \left( \left| n^{-1/2} \sum_{i=1}^n (A_i - \mathbb{E}[A_i | \mathbf{U}_n, \mathbf{E}]) \right| \geq 8 \max_i \mathbb{E}[N_i | \mathbf{U}]^{-1/2} \sqrt{t} + 8C_{\beta,h} n^{-1/2} t \mid \mathbf{U} \right) \leq \exp(-t) + n^{-99},$$

that is

$$n^{-\mathbf{a}_{\beta,h}} \sum_{i=1}^n (A_i - \mathbb{E}[A_i | \mathbf{U}_n, \mathbf{E}]) = O_{\psi_2,tc} \left( n^{\frac{1}{2} - \mathbf{a}_{\beta,h}} \max \mathbb{E}[N_i | \mathbf{U}]^{-1/2} \right) + O_{\psi_1,tc} \left( n^{-1/2} \right). \quad (\text{SA-20})$$

**Second part of  $\Delta_{2,3,1,a}$ :** Next, we will show  $n^{1-\mathbf{a}_{\beta,h}} (\mathbb{E}[A_i | \mathbf{U}, \mathbf{U}, \mathbf{E}] - \mathbb{E}[A_i | \mathbf{E}])$ , is small. There exists a function  $F$  that possibly depends on  $\beta$  and  $\mathbf{E}$  such that

$$F(\mathbf{W}, \mathbf{U}_n) = \frac{1}{2} \theta \left( \frac{M_i}{N_i} \right) \left( \sum_{j \neq i} \frac{E_{ij}}{N_i} (W_j - \mathbb{E}[W_j | \mathbf{U}_n]) \right)^2.$$

Define  $p(u) = \mathbb{P}(W_j = 1|U, \mathbf{U})$ . Then

$$\mathbb{E}[A_i|U = u, \mathbf{U}, \mathbf{E}] = \mathbb{E}[F(\mathbf{W}, U)|U = u] = \sum_{\mathbf{w} \in \{-1,1\}^n} \prod_{l=1}^n p(u)^{w_l} (1-p(u))^{1-w_l} F(\mathbf{w}, u).$$

Using chain rule and product rule for derivatives,

$$\begin{aligned} & \partial_u \mathbb{E}[A_i|U = u, \mathbf{U}] \\ &= \sum_{\mathbf{w} \in \{-1,1\}^n} \left[ \sum_{l=1}^n \prod_{s \neq l} p(u)^{w_s} (1-p(u))^{1-w_s} (F((\mathbf{w}_{-l}, w_l = 1), u) - F((\mathbf{w}_{-l}, w_l = -1), u)) \right. \\ & \quad \left. + \prod_{i=1}^n p(u)^{w_i} (1-p(u))^{1-w_i} \partial_u F(\mathbf{w}, u) \right] p'(u) \\ &= \sum_{l=1}^n \mathbb{E}_{\mathbf{W}_{-l}} [F((\mathbf{W}_{-l}, W_l = 1), u) - F((\mathbf{W}_{-l}, W_l = -1), u)] p'(u) + \mathbb{E}_{\mathbf{W}} [\partial_u F(\mathbf{W}, u)] p'(u) \\ &= \sum_{l=1}^n O_{\mathbb{P}} \left( \frac{1}{\sqrt{N_i}} \frac{E_{il}}{N_i} \right) \|p'\|_{\infty} + O_{\mathbb{P}} \left( \frac{1}{\sqrt{N_i}} \|p'\|_{\infty} \right) \|p'\|_{\infty} = O_{\mathbb{P}}((nN_i)^{-0.5}), \end{aligned}$$

where in the last line, we have used

$$\begin{aligned} |D_{W_l} F(\mathbf{w}, u)| &\lesssim \|\theta'\|_{\infty} \frac{E_{il}}{N_i} \left( \sum_{j \neq i} \frac{E_{ij}}{N_i} (W_j - \mathbb{E}[W_j|U, \mathbf{U}]) \right)^2 + \|\theta\|_{\infty} \left| \sum_{j \neq i} \frac{E_{ij}}{N_i} (W_j - \mathbb{E}[W_j|U, \mathbf{U}]) \right| \frac{E_{il}}{N_i}, \\ |\partial_u F(\mathbf{w}, u)| &\lesssim \|\theta\|_{\infty} \|p'\|_{\infty} \left| \sum_{j \neq i} \frac{E_{ij}}{N_i} (W_j - \mathbb{E}[W_j|U, \mathbf{U}]) \right|, \end{aligned}$$

and that fact that  $\|p'\|_{\infty} = O((2\beta/n)^{0.5})$  and Hoeffding's inequality for  $\sum_{j \neq i} \frac{E_{ij}}{N_i} (W_j - \mathbb{E}[W_j|U_n])$ ,

$$|\partial_u \mathbb{E}[F(\mathbf{w}, \mathbf{U}_n)|\mathbf{U}_n = u, \mathbf{E}]| \leq \mathbb{E} [|\partial_u F(\mathbf{w}, \mathbf{U}_n)||\mathbf{U}_n = u] = O \left( n^{-1/2} \min_i N_i^{-1/2} \right).$$

Since  $\mathbf{U}_n = O_{\psi_{\beta,h}}(n^{\mathbf{a}_{\beta,h}-1/2})$ , we have

$$\begin{aligned} n^{-\mathbf{a}_{\beta,h}} \sum_{i=1}^n (\mathbb{E}[A_i|\mathbf{U}_n, \mathbf{U}] - \mathbb{E}[A_i|\mathbf{U}]) &= O_{\psi_{\beta,h}} \left( n^{1-\mathbf{a}_{\beta,h}} n^{-1/2} \min_i N_i^{-1/2} n^{\mathbf{a}_{\beta,h}-1/2} \right) \\ &= O_{\psi_{\beta,h}} \left( \min_i N_i^{-1/2} \right). \end{aligned} \tag{SA-21}$$

Combining Equations SA-20 and SA-21, conditional on  $\mathbf{U}$  such that  $A(\mathbf{U}) \in \mathcal{A}$ ,

$$\begin{aligned} n^{-\mathbf{a}_{\beta,h}} \sum_{i=1}^n (A_i - \mathbb{E}[A_i|\mathbf{E}]) &= O_{\psi_{2,tc}} \left( n^{\frac{1}{2}-\mathbf{a}_{\beta,h}} \max \mathbb{E}[N_i|\mathbf{U}]^{-1/2} \right) + O_{\psi_{1,tc}} \left( n^{-1/2} \right) \\ & \quad + O_{\psi_{\beta,h,tc}} \left( \max_i \mathbb{E}[N_i]^{-1/2} \right). \end{aligned}$$

Combining the bounds for  $\Delta_{2,3,1,a}, \Delta_{2,3,1,b}, \Delta_{2,3,1,c}$ , we get the desired result.

### SA-7.3.6 Proof of Lemma SA-5

Recall

$$\Delta_{2,3,2} = n^{-\mathbf{a}_{\beta,h}} \sum_{i=1}^n \frac{1}{2} \frac{W_i - \mathbb{E}[W_i | \mathbf{W}_{-i}]}{p_i} \left[ g_i \left( 1, \frac{M_i}{N_i} \right) - g_i(1, \pi) - g'_i(1, \pi) \left( \frac{M_i}{N_i} - \pi \right) \right].$$

First, we will consider the effect of fluctuation of  $p_i$  and  $\mathbb{E}[W_i | \mathbf{W}_{-i}]$ . Recall

$$\mathbb{E}[W_i | \mathbf{W}_{-i}] = \tanh(\beta m_i + h), \quad p_i = (1 + \exp(-2\beta m_i - 2h))^{-1}.$$

It follows from the boundeness of  $\beta m_i + h$ ,  $m_i - \pi = O_{\psi_{\beta,h}}(n^{-\mathbf{r}_{\beta,h}})$  that for each  $i \in [n]$ ,

$$\frac{W_i - \mathbb{E}[W_i | \mathbf{W}_{-i}]}{p_i} = 2 \frac{W_i - \pi}{\pi + 1} + O_{\psi_{\beta,h}}(n^{-\mathbf{r}_{\beta,h}}).$$

Moreover for some  $\eta_i^*$  between  $M_i/N_i$  and  $\pi$ , using Lemma SA-2 we have

$$\begin{aligned} & g_i \left( 1, \frac{M_i}{N_i} \right) - g_i(1, \pi) - g'_i(1, \pi) \left( \frac{M_i}{N_i} - \pi \right) \\ &= \frac{1}{2} g_i''(1, \eta_i^*) \left( \frac{M_i}{N_i} - \pi \right)^2 = O_{\psi_{\mathbf{p}_{\beta,h}/2,tc}}(n^{-2\mathbf{r}_{\beta,h}}) + O_{\psi_{1,tc}}(N_i^{-1}). \end{aligned}$$

Using a union bound over  $i$  and an argument for the product of two terms with bounded Orlicz norm with tail control, we have

$$\begin{aligned} \Delta_{2,3,2} &= n^{-\mathbf{a}_{\beta,h}} \sum_{i=1}^n \frac{W_i - \pi}{\pi + 1} \left[ g_i \left( 1, \frac{M_i}{N_i} \right) - g_i(1, \pi) - g'_i(1, \pi) \left( \frac{M_i}{N_i} - \pi \right) \right] \\ &\quad + O_{\psi_{\mathbf{p}_{\beta,h}/2,tc}}((\log n)^{-1/\mathbf{p}_{\beta,h}} n^{-2\mathbf{r}_{\beta,h}}) + O_{\psi_{1,tc}}((\log n)^{-1/\mathbf{p}_{\beta,h}} N_i^{-1}). \end{aligned}$$

Next, we will show  $n^{-\mathbf{a}_{\beta,h}} \sum_{i=1}^n \frac{W_i - \pi}{\pi + 1} \left[ g_i \left( 1, \frac{M_i}{N_i} \right) - g_i(1, \pi) - g'_i(1, \pi) \left( \frac{M_i}{N_i} - \pi \right) \right]$  is small. Suppose  $g_i(1, \cdot)$  is  $p$ -times continuously differentiable. Define

$$\delta_p = n^{-\mathbf{a}_{\beta,h}} \sum_{i=1}^n \frac{W_i - \pi}{\pi + 1} g_i^{(p)}(1, \pi) \left( \frac{M_i}{N_i} - \pi \right)^p.$$

We will use the conditioning strategy to analyse  $\delta_p$ : Decompose by

$$\delta_p = \delta_{p,1} + \delta_{p,2} + \delta_{p,3},$$

with

$$\begin{aligned} \delta_{p,1} &= n^{-\mathbf{a}_{\beta,h}} \sum_{i=1}^n \frac{W_i - \mathbb{E}[W_i | \mathbf{U}_n]}{\pi + 1} g_i^{(p)}(1, \pi) \left( \frac{M_i}{N_i} - \mathbb{E}[W_i | \mathbf{U}_n] \right)^p, \\ \delta_{p,2} &= n^{-\mathbf{a}_{\beta,h}} \sum_{i=1}^n \frac{\mathbb{E}[W_i | \mathbf{U}_n] - \pi}{\pi + 1} g_i^{(p)}(1, \pi) \left( \frac{M_i}{N_i} - \mathbb{E}[W_i | \mathbf{U}_n] \right)^p, \\ \delta_{p,3} &= n^{-\mathbf{a}_{\beta,h}} \sum_{i=1}^n \frac{W_i - \pi}{\pi + 1} g_i^{(p)}(1, \pi) \left[ \left( \frac{M_i}{N_i} - \mathbb{E}[W_i | \mathbf{U}_n] \right)^p - \left( \frac{M_i}{N_i} - \pi \right)^p \right]. \end{aligned}$$

First, we will show  $\delta_{p,2}$  and  $\delta_{p,3}$  are small. By Hoeffding inequality,  $M_i/N_i - \mathbb{E}[W_i|\mathbf{U}_n] = O_{\psi_2}(N_i^{-1/2})$ . Moreover,  $\mathbb{E}[W_i|\mathbf{U}_n] - \pi = O_{\psi_{\beta,h}}(n^{-\mathbf{r}_{\beta,h}})$ . Hence

$$\delta_{p,2} = O_{\psi_{\beta,h,tc}}(\max_i N_i^{-1/2}).$$

For  $\delta_{p,3}$ , we have

$$\left(\frac{M_i}{N_i} - \mathbb{E}[W_i|\mathbf{U}_n]\right)^p - \left(\frac{M_i}{N_i} - \pi\right)^p = p \left(\frac{M_i}{N_i} - \xi^*\right)^{p-1} (\mathbb{E}[W_i|\mathbf{U}_n] - \pi),$$

where  $\xi^*$  is some quantity between  $\mathbb{E}[W_i|\mathbf{U}_n]$  and  $\pi$ . Since  $x \mapsto x^{p-1}$  is either monotone or convex and non-negative, condition on  $\mathbf{E}$ ,

$$\begin{aligned} \left|\frac{M_i}{N_i} - \xi^*\right|^{p-1} &\leq \max \left\{ \left|\frac{M_i}{N_i} - \mathbb{E}[W_i|\mathbf{U}_n]\right|^{p-1}, \left|\frac{M_i}{N_i} - \pi\right|^{p-1} \right\} \\ &= O_{\psi_{\frac{p\beta,h}{p-1}}}(n^{-(p-1)\mathbf{r}_{\beta,h}}) + O_{\psi_{\frac{2}{p-1}}}(N_i^{-\frac{p-1}{2}}). \end{aligned}$$

Combining with boundedness of  $g_i^{(p)}(1, \pi)$  and tail control of  $\mathbb{E}[W_i|\mathbf{U}_n]$ , we have

$$\delta_{p,3} = O_{\psi_{\frac{p\beta,h}{p-1}}} \left( (\log n)^{\frac{1}{p\beta,h}} n^{-(p-1)\mathbf{r}_{\beta,h}} \right) + O_{\psi_{\frac{2}{p-1}}} \left( (\log n)^{\frac{1}{p\beta,h}} N_i^{-\frac{p-1}{2}} \right).$$

For  $\delta_{p,1}$ , we will again use the generalized version of Hanson-Wright inequality. For each  $k \in [n]$ ,

$$\begin{aligned} \partial_k \delta_{p,1} &= n^{-\mathbf{a}_{\beta,h}} \sum_{i \neq k} \frac{W_i - \mathbb{E}[W_i|\mathbf{U}_n]}{\pi + 1} g_i^{(p)}(1, \pi) p \left(\frac{M_i}{N_i} - \mathbb{E}[W_i|\mathbf{U}_n]\right)^{p-1} \frac{E_{ik}}{N_i} \\ &\quad + n^{-\mathbf{a}_{\beta,h}} g_k^{(p)}(1, \pi) \left(\frac{M_k}{N_k} - \mathbb{E}[W_i|\mathbf{U}_n]\right)^p. \end{aligned}$$

Hence condition on  $\mathbf{E}$ ,

$$\|\mathbb{E}[\nabla \delta_{p,1}]\| = O \left( n^{1/2 - \mathbf{a}_{\beta,h}} N_i^{-(p-1)/2} \right).$$

Taking mixed partials w.r.p  $\delta_{p,1}$  and using boundedness of  $g_i^{(p)}$ , we have

$$\|\partial_k \partial_l \delta_{p,1}\|_{\infty} \lesssim n^{-\mathbf{a}_{\beta,h}} \sum_{i \neq k,l} \frac{E_{ik} E_{il}}{N_i^2} + n^{-\mathbf{a}_{\beta,h}} \frac{E_{lk}}{N_l} + n^{-\mathbf{a}_{\beta,h}} \frac{E_{kl}}{N_k}.$$

It follows that

$$\|\|\text{Hess}(\delta_{p,1})\|_2\|_{\infty} \lesssim \|\|\text{Hess}(\delta_{p,1})\|_F\|_{\infty} \lesssim n^{1/2 - \mathbf{a}_{\beta,h}} \left( \frac{\max_i N_i^3}{\min_i N_i^4} \right)^{1/2}.$$

It then follows from Equation SA-18 and Theorem 3 in [4] that conditional on  $\mathbf{U}$  such that  $A(\mathbf{U}) \in \mathcal{A}$ ,

$$\delta_{p,1} - \mathbb{E}[\delta_{p,1}|\mathbf{E}] = O_{\psi_{1,tc}} \left( n^{1/2 - \mathbf{a}_{\beta,h}} \left( \frac{\max_i \mathbb{E}[N_i|\mathbf{U}]^3}{\min_i \mathbb{E}[N_i|\mathbf{U}]^4} \right)^{1/2} \right).$$

**Trade-off Between Smoothness of  $g_i(1, \cdot)$  and Sparsity of Graph** Assume  $g_i(1, \cdot)$  is  $p + 1$ -times continuously differentiable. Then by the decomposition of  $\Delta_{2,3,2}$ , condition on  $\mathbf{U}$  such that  $A(\mathbf{U}) \in \mathcal{A}$ ,

$$\begin{aligned} & \Delta_{2,3,2} - \mathbb{E}[\Delta_{2,3,2} | \mathbf{E}] \\ &= \sum_{l=2}^p \delta_l - \mathbb{E}[\delta_l | \mathbf{E}] + n^{-\mathbf{a}_{\beta,h}} \sum_{i=1}^n \left[ \frac{Y_i^{(p+1)}(1, \xi_i^*)}{(p+1)!} \left( \frac{M_i}{N_i} - \pi \right)^{p+1} - \mathbb{E} \left[ \frac{Y_i^{(p+1)}(1, \xi_i^*)}{(p+1)!} \left( \frac{M_i}{N_i} - \pi \right)^{p+1} \middle| \mathbf{E} \right] \right] \\ & \quad + O_{\psi_{p_{\beta,h}/2,tc}}((\log n)^{-1/p_{\beta,h}} n^{-2\mathbf{r}_{\beta,h}}) + O_{\psi_{1,tc}}((\log n)^{-1/p_{\beta,h}} (\min_i \mathbb{E}[N_i | \mathbf{U}])^{-1}). \end{aligned}$$

Then by the concentration of  $M_i/N_i - \pi$  given in Lemma SA-2, we have

$$\begin{aligned} & \Delta_{2,3,2} - \mathbb{E}[\Delta_{2,3,2} | \mathbf{E}] \\ &= O_{\psi_{p_{\beta,h}/2,tc}}((\log n)^{-1/p_{\beta,h}} n^{-2\mathbf{r}_{\beta,h}}) + O_{\psi_{1,tc}}((\log n)^{-1/p_{\beta,h}} (\min_i \mathbb{E}[N_i | \mathbf{U}])^{-1}) \\ & \quad + O_{\psi_{1,tc}} \left( n^{1/2-\mathbf{a}_{\beta,h}} \left( \frac{\max_i \mathbb{E}[N_i | \mathbf{U}]^3}{\min_i \mathbb{E}[N_i | \mathbf{U}]^4} \right)^{1/2} \right) + O_{\psi_{2/(p+1),tc}} \left( n^{\mathbf{r}_{\beta,h}} (\min_i \mathbb{E}[N_i | \mathbf{U}]^{-(p+1)/2}) \right). \end{aligned}$$

### SA-7.3.7 Proof of Lemma SA-6

For notational simplicity, denote  $\hat{p} = \frac{1}{n} \sum_{i=1}^n T_i$  and  $p = \frac{1}{2} \tanh(\beta\pi + h) + \frac{1}{2} = \frac{1}{2}\pi + \frac{1}{2}$ . Then

$$\frac{1}{n} \sum_{i=1}^n \frac{T_i Y_i}{\hat{p}} - \frac{1}{n} \sum_{i=1}^n \frac{T_i Y_i}{p} = \frac{1}{n} \sum_{i=1}^n \frac{T_i Y_i}{\hat{p}} \frac{p - \hat{p}}{p}.$$

Taylor expand  $x \mapsto \tanh(\beta x + h)$  at  $x = \pi$ , we have

$$\begin{aligned} 2(\hat{p} - p) &= m - \tanh(\beta m + h) \\ &= \pi + m - \pi - \tanh(\beta\pi + h) - \beta \operatorname{sech}^2(\beta\pi + h)(m - \pi) + O((m - \pi)^2) \\ &= (1 - \beta \operatorname{sech}^2(\beta\pi + h))(m - \pi) + O((m - \pi)^2), \end{aligned}$$

where  $O(\cdot)$  is up to a universal constant. Together with concentration of  $\frac{1}{n} \sum_{i=1}^n T_i Y_i$  towards  $p\mathbb{E}[Y_i]$ , we have

$$\frac{1}{n} \sum_{i=1}^n \frac{T_i Y_i}{\hat{p}} - \frac{1}{n} \sum_{i=1}^n \frac{T_i Y_i}{p} = -\frac{1 - \beta(1 - \pi^2)}{1 + \pi} \mathbb{E}[Y_i(1, \frac{M_i}{N_i})] + O_{\psi_1}(n^{-2\mathbf{r}_{\beta,h}}).$$

### SA-7.3.8 Proof of Lemma SA-7

By Lemma SA-2 to Lemma SA-6, we show

$$n^{\mathbf{r}_{\beta,h}} (\hat{\tau}_n - \tau_n) \tag{SA-22}$$

$$= n^{-\mathbf{a}_{\beta,h}} \sum_{i=1}^n (R_i - \mathbb{E}[R_i] + b_i)(W_i - \pi) + \varepsilon, \tag{SA-23}$$

where  $R_i = \frac{g_i(1, \frac{M_i}{N_i})}{1+\pi} + \frac{g_i(-1, \frac{M_i}{N_i})}{1-\pi}$ , and  $b_i = \sum_{j \neq i} \frac{E_{ij}}{N_j} g'_j(1, \pi)$ , and  $\varepsilon$  is such that condition on  $\mathbf{U}$  such that  $A(\mathbf{U}) \in \mathcal{A} = \{A \in \mathbb{R}^{n \times n} : \min_{i \in [n]} \sum_{j \neq i} A_{ij} \geq 32 \log n\}$ ,

$$\begin{aligned} \varepsilon &= O_{\psi_1, tc} \left( \log n \max_{i \in [n]} \mathbb{E}[N_i | \mathbf{U}]^{-1/2} \right) + O_{\psi_1, tc}(\sqrt{\log n} n^{-r_{\beta, h}}) \\ &+ O_{\psi_1, tc} \left( n^{1/2 - a_{\beta, h}} \left( \frac{\max_i \mathbb{E}[N_i | \mathbf{U}]^3}{\min_i \mathbb{E}[N_i | \mathbf{U}]^4} \right)^{1/2} \right) + O_{\psi_2/(p+1), tc} \left( n^{r_{\beta, h}} (\min_i \mathbb{E}[N_i | \mathbf{U}]^{-(p+1)/2}) \right). \end{aligned} \quad (\text{SA-24})$$

Following the strategy as in the proof of Theorem 4 in [6], we will show  $b_i$  is close to  $R_i$ : First, decompose by

$$\begin{aligned} &\sum_{j \neq i} \frac{E_{ij}}{N_j} g'_j(1, \pi) - R_i \\ &= \sum_{j \neq i} \frac{E_{ij}}{N_j} g'_j(1, \pi) - \sum_{j \neq i} \frac{E_{ij}}{n \mathbb{E}[G(U_i, U_j) | U_j]} g'_j(1, \pi) + \sum_{j \neq i} \frac{E_{ij}}{n \mathbb{E}[G(U_i, U_j) | U_j]} g'_j(1, \pi) - R_i. \end{aligned}$$

By Equation SA-18, condition on  $\mathbf{U}$  such that  $A(\mathbf{U}) \in \mathcal{A}$ ,

$$\left| \sum_{j \neq i} \frac{E_{ij}}{N_j} g'_j(1, \pi) - \sum_{j \neq i} \frac{E_{ij}}{n \mathbb{E}[G(U_i, U_j) | U_j]} g'_j(1, \pi) \right| \leq C n^{-1/2}$$

with probability at least  $1 - n^{-99}$ . Moreover,  $\frac{E_{ij}}{\mathbb{E}[G(U_i, U_j) | U_j]} g'_j(1, \pi), j \neq i$  are i.i.d condition on  $U_i$ , hence  $\sum_{j \neq i} \frac{E_{ij}}{n \mathbb{E}[G(U_i, U_j) | U_j]} g'_j(1, \pi) - R_i = O_{\psi_2}((n \mathbb{E}[G(U_i, U_j) | U_j]^{-1/2}) = O_{\psi_2}(\mathbb{E}[N_j | X]^{-1/2})$ . It follows that conditional on  $\mathbf{U}$  such that  $A(\mathbf{U}) \in \mathcal{A}$ ,

$$\max_i \left| \sum_{j \neq i} \frac{E_{ij}}{N_j} g'_j(1, \pi) - R_i \right| = O_{\psi_2, tc}(\max_i \mathbb{E}[N_i | \mathbf{U}]^{-1/2}). \quad (\text{SA-25})$$

Again using the conditional i.i.d decomposition, Hoeffding inequality and  $U_n$ 's concentration for the two terms respectively,

$$\begin{aligned} &|n^{-a_{\beta, h}} \sum_{i=1}^n [\sum_{j \neq i} \frac{E_{ij}}{N_j} g'_j(1, \pi) - R_i] (W_i - \pi)| \\ &\leq |n^{-a_{\beta, h}} \sum_{i=1}^n [\sum_{j \neq i} \frac{E_{ij}}{N_j} g'_j(1, \pi) - R_i] (W_i - \mathbb{E}[W_i | U_n])| \\ &\quad + n^{r_{\beta, h}} |\mathbb{E}[W_i | U_n] - \pi| \max_i \left| \sum_{j \neq i} \frac{E_{ij}}{N_j} g'_j(1, \pi) - R_i \right| \\ &= O_{\psi_2}(n^{\frac{1}{2} - a_{\beta, h}} \max_i \mathbb{E}[N_i | \mathbf{U}]^{-1/2}) + O_{\psi_{\beta, h}, tc}((\log n)^{1/p_{\beta, h}} \max_i \mathbb{E}[N_i | \mathbf{U}]^{-1/2}) = \varepsilon'. \end{aligned}$$

Hence denote the term of stochastic linearization by  $G_n$ , i.e.

$$G_n = n^{-a_{\beta, h}} \sum_{i=1}^n (R_i - \mathbb{E}[R_i] + Q_i) (W_i - \pi).$$

Since  $R_i - \mathbb{E}[R_i] + Q_i$ 's are i.i.d independent to  $W_i$ 's with bounded third moment, we know from Lemma SA-3 that  $G_n$  can be approximated by either a Gaussian or non-Gaussian law, that is order 1, this gives

$$\begin{aligned}
& \sup_{t \in \mathbb{R}} \mathbb{P}(\widehat{\tau}_n - \tau_n | \mathbf{U}) \leq t) - \mathbb{P}(G_n \leq t | \mathbf{U}) \\
& \leq \sup_{t \in \mathbb{R}} \min_{u > 0} \mathbb{P}(G_n \leq t + u) - \mathbb{P}(G_n \leq t) + \mathbb{P}(\varepsilon + \varepsilon' \geq u) \\
& \leq \sup_{t \in \mathbb{R}} \min_{u > 0} \mathbb{P}(G_n \leq t + u) - \mathbb{P}(G_n \leq t + u) + \mathbb{P}(\varepsilon + \varepsilon' \geq u) + \mathbb{P}(t \leq G_n \leq t + u) \\
& \leq O(n^{-1/2}) + \min_{u > 0} \exp(-(u/\mathbf{r})^a) + cu \\
& = O((\log n)^a \mathbf{r}(\mathbf{U})),
\end{aligned}$$

where  $O(\cdot)$  does not depend on the value of  $\mathbf{U}$  and

$$\begin{aligned}
\mathbf{r}(\mathbf{U}) &= n^{-\mathbf{r}\beta, h} + \max_i \mathbb{E}[N_i | \mathbf{U}]^{-1/2} + n^{1/2 - \mathbf{a}\beta, h} \left( \frac{\max_i \mathbb{E}[N_i | \mathbf{U}]^3}{\min \mathbb{E}[N_i | \mathbf{U}]^4} \right)^{1/2} \\
&\quad + n^{\mathbf{r}\beta, h} \max_i \mathbb{E}[N_i | \mathbf{U}]^{-(p+1)/2}.
\end{aligned}$$

To analyse the second term, recall  $\mathbb{E}[N_i | \mathbf{U}] = \rho_n \sum_{j \neq i} G(U_i, U_j)$ . Hence

$$\begin{aligned}
& \mathbb{E} \left[ \max_i (\mathbb{E}[N_i | \mathbf{U}])^{-1/2} \mathbb{1}(A(\mathbf{U}) \in \mathcal{A}) \right] \\
&= (n\rho_n)^{-1/2} \mathbb{E} \left[ \max_i \left( \frac{1}{n} \sum_{j \neq i} G(U_i, U_j) \right)^{-1/2} \mathbb{1}(A(\mathbf{U}) \in \mathcal{A}) \right] \\
&= O(\sqrt{\log n} (n\rho_n)^{-1/2}),
\end{aligned}$$

the last line is because with probability at least  $1 - n^{-98}$ ,  $E = \{\frac{1}{2}g(U_i) \leq \frac{1}{n} \sum_{j \neq i} G(U_i, U_j) \leq 2g(U_i), \forall 1 \leq i \leq n\}$  happens, and by maximal inequality,  $\max_i |g(U_i)|^{-1/2} = O_{\psi_2}(\sqrt{\log n})$ . And on  $\{A(\mathbf{U}) \in \mathcal{A} \cap E\}$ ,  $\max_i (\frac{1}{n} \sum_{j \neq i} G(U_i, U_j))^{-1/2} \leq (32 \log n/n)^{-1/2}$ , since we assume  $G$  is positive. By similar argument for the last two terms in  $\mathbf{r}(\mathbf{U})$ , we have

$$\mathbb{E}[r(\mathbf{U}) \mathbb{1}(A(\mathbf{U}) \in \mathcal{A})] \leq n^{-\mathbf{r}\beta, h} + \sqrt{\log n} (n\rho_n)^{-1/2} + \sqrt{\log n} n^{\mathbf{r}\beta, h} (n\rho_n)^{-(p+1)/2}.$$

Recall that  $\mathcal{A} = \{A(\mathbf{U}) : \min_i \sum_{j \neq i} A_{ij}(\mathbf{U}) \geq 32 \log n\}$ . Since  $\sum_{j \neq i} A_{ij}(\mathbf{U}) \sim \text{Bin}(n-1, \mathbb{E}[G(X_1, X_2)])$ , we know from Chernoff bound for Binomials and union bound over  $i$  that  $\mathbb{P}(A(\mathbf{U}) \notin \mathcal{A}) \leq n^{-99}$ . The conclusion then follows.

### SA-7.3.9 Proof of Lemma SA-8

Our proof for Lemma SA-2 to Lemma SA-6 relies on the following devices:

(1) Taylor expansion of  $\tanh(\cdot)$  in the inverse probability weighting for unbiased estimator, and Taylor expansion of  $Y_i(\ell, \cdot)$  at  $\mathbb{E}[T_i]$  for  $\ell \in \{0, 1\}$ . Then the higher order terms are in terms of  $m - \pi$  and  $\frac{M_i}{N_i} - \pi$ . In Lemma SA-4 (taking  $X_i \equiv 1$ ), we show

$$\|m\|_{\psi_1} \leq Kn^{-1/4},$$

and in Lemma SA-2, we show

$$\left\| \frac{M_i}{N_i} \right\|_{\psi_1} \leq Kn^{-1/4} + K(n\rho_n)^{-1/2},$$

where  $K$  is some constant that does not depend on  $\beta$ . This shows for the higher order terms, we always have

$$m^2 = m(1 + o_{\mathbb{P}}(1)), \quad (M_i/N_i)^2 = (M_i/N_i)(1 + o_{\mathbb{P}}(1)),$$

where the  $o_{\mathbb{P}}(\cdot)$  terms does not depend on  $\beta$ .

(2) Condition i.i.d decomposition based on the de-Finetti's lemma (Lemma SA-1). Suppose  $\mathbf{U}_n$  is the latent variable from Lemma SA-1, we use decompositions based on  $\mathbf{U}_n$ : For Lemma SA-3 to Lemma SA-5, we break down higher order terms in the form

$$\begin{aligned} & F(\mathbf{W}, \mathbf{E}) - \mathbb{E}[F(\mathbf{W}, \mathbf{E})|\mathbf{E}] \\ &= F(\mathbf{W}, \mathbf{E}) - \mathbb{E}[F(\mathbf{W}, \mathbf{E})|\mathbf{E}, \mathbf{U}_n] + \mathbb{E}[F(\mathbf{W}, \mathbf{E})|\mathbf{E}, \mathbf{U}_n] - \mathbb{E}[F(\mathbf{W}, \mathbf{E})|\mathbf{E}]. \end{aligned}$$

For the first part  $F(\mathbf{W}, \mathbf{E}) - \mathbb{E}[F(\mathbf{W}, \mathbf{E})|\mathbf{E}, \mathbf{U}_n]$ , we use the conditional i.i.d of  $W_i$ 's given  $\mathbf{U}_n$ . For the second part, we use concentration from Lemma SA-2 that there exists a constant  $K$  not depending on  $\beta$  or  $n$ , such that  $\|\mathbf{U}_n\|_{\psi_1} \leq Kn^{1/4}$  and the effective term  $\|\tanh(\sqrt{\frac{\beta}{n}}\mathbf{U}_n)\|_{\psi_1} \leq Kn^{-1/4}$ .

In particular, the rate of concentration for conditional i.i.d Berry-Esseen and concentration of  $\tanh(\sqrt{\frac{\beta}{n}}\mathbf{U}_n)$  does not depend on  $\beta$ .

By the same proof from Lemma SA-2 to Lemma SA-6, we can show in  $\hat{\tau}_n - \tau_n$ , the second and higher order terms in terms of  $W_i - \pi$  can always be dominated by the first order terms, with a rate that does not depend on  $\beta$ .

The conclusion then follows from the two devices and the same proof logic of Lemma SA-2 to Lemma SA-6.

## SA-7.4 Proof for Section SA-5

### SA-7.4.1 Proof of Lemma SA-1

Define  $g(U_j) = \mathbb{E}[G(U_i, U_j)|U_j]$ , for  $i \neq j$ . Reordering the terms,

$$\bar{\tau}^a = \frac{n-1}{n^2} \sum_{j \in [n]} \frac{T_j}{1/2} h_j(1, M_j/N_j) - \frac{1-T_j}{1-1/2} h_j(-1, M_j/N_j).$$

Hence  $\tau_{(i)}^a - \bar{\tau}^a$  has the representation given by

$$\begin{aligned} & \tau_{(i)}^a - \bar{\tau}^a \\ &= -\frac{1}{n} \frac{T_i}{1/2} h_i\left(1, \frac{M_i}{N_i}\right) + \frac{1}{n^2} \sum_{j \in [n]} \frac{T_j}{1/2} h_j\left(1, \frac{M_j}{N_j}\right) + \frac{1}{n} \frac{1 - T_i}{1 - 1/2} h_i\left(1, \frac{M_i}{N_i}\right) \\ & \quad - \frac{1}{n^2} \sum_{j \in [n]} \frac{1 - T_j}{1 - 1/2} h_j\left(1, \frac{M_j}{N_j}\right) \end{aligned} \quad (\text{SA-26})$$

$$\begin{aligned} &= -\frac{1}{n} \left( \frac{T_i}{1/2} h_i(1, 0) - 1/2 \mathbb{E}[h_i(1, 0)] \right) + \frac{1}{n} \left( \frac{1 - T_i}{1 - 1/2} h_i(-1, 0) - (1 - 1/2) \mathbb{E}[h_i(-1, 0)] \right) \\ & \quad + O_{\psi_{2,tc}}(n^{-1}(n\rho_n)^{-\frac{1}{2}}) \end{aligned} \quad (\text{SA-27})$$

$$= -\frac{1}{n} \left( \frac{h_i(1, 0)}{1/2} + \frac{h_i(-1, 0)}{1 - 1/2} \right) (T_i - 1/2) + O_{\psi_{2,tc}}(n^{-1}(n\rho_n)^{-\frac{1}{2}}) \quad (\text{SA-28})$$

$$= -\frac{1}{n} \left( \frac{f_i(1, 0)}{1/2} + \frac{f_i(-1, 0)}{1 - 1/2} \right) (T_i - 1/2) + O_{\psi_{2,tc}}(n^{-1}(n\rho_n)^{-\frac{1}{2}}) + o_{\mathbb{P}}(n^{-1}), \quad (\text{SA-29})$$

where the second to last line is due to  $-\frac{1}{n} \frac{1}{1/2} 1/2 (h_i(1, 0) - \mathbb{E}[h_i(1, 0)]) + \frac{1}{n} \frac{1}{1 - 1/2} (1 - 1/2) (h_i(-1, 0) - \mathbb{E}[h_i(-1, 0)]) = -\frac{2}{n} \varepsilon_i + \frac{2}{n} \varepsilon_i = 0$ .

Now we look at  $b$ -part. For representation purpose, we look at only the treatment part. The control part can be analyzed by in the same way. Reordering the terms,

$$\begin{aligned} \bar{\tau}^b &= \frac{1}{n} \sum_{i \in [n]} \tau_{(i)}^b = \frac{1}{n} \sum_{i \in [n]} \frac{1}{n} \sum_{j \in [n]} \frac{T_j}{1/2} \left[ h_j\left(1, \frac{M_j}{N_j^{(i)}}\right) - h_j\left(1, \frac{M_j}{N_j}\right) \right] \\ &= \frac{1}{n} \sum_{j \in [n]} \frac{T_j}{1/2} \frac{1}{n} \sum_{i \in [n]} \left[ h_j\left(1, \frac{M_j}{N_j^{(i)}}\right) - h_j\left(1, \frac{M_j}{N_j}\right) \right]. \end{aligned}$$

Hence  $\tau_{(i)}^b - \bar{\tau}^b$  has the representation given by

$$\tau_{(i)}^b - \bar{\tau}^b = \frac{1}{n} \sum_{j \in [n]} \frac{T_j}{1/2} \left[ h_j\left(1, \frac{M_j}{N_j^{(i)}}\right) - \frac{1}{n} \sum_{\iota \in [n]} h_j\left(1, \frac{M_j}{N_j^{(\iota)}}\right) \right]. \quad (\text{SA-30})$$

The analysis follows from a Taylor expansion of  $h_j(1, \cdot)$ . For some  $\xi_{j,i}^*$  between  $\frac{M_j}{N_j^{(i)}}$  and 0 for each  $j, i$ ,

$$h_j\left(1, \frac{M_j}{N_j^{(i)}}\right) = h_j(1, 0) + \partial_2 h(1, 0) \left( \frac{M_j}{N_j^{(i)}} - 0 \right) + \frac{1}{2} \partial_{2,2} h(1, 0) \left( \frac{M_j}{N_j^{(i)}} - 0 \right)^2 \quad (\text{SA-31})$$

$$+ \frac{1}{6} \partial_{2,2,2} h(1, \xi_{j,i}^*) \left( \frac{M_j}{N_j^{(i)}} - 0 \right)^3, \quad (\text{SA-32})$$

where we have used  $\partial_2 h_j(1, \cdot) = \partial_2 [h(1, \cdot) + \varepsilon_j] = \partial_2 h(1, \cdot)$ .

### Part 1: Linear Terms

$$\begin{aligned} \frac{M_j}{N_j^{(i)}} - \frac{1}{n} \sum_{\iota \in [n]} \frac{M_j}{N_j^{(\iota)}} &= \sum_{l \neq i} \frac{E_{lj}}{N_j^{(i)}} W_l - \frac{1}{n} \sum_{\iota \in [n]} \sum_{l \neq \iota} \frac{E_{lj}}{N_j^{(\iota)}} W_l \\ &= \sum_{l=1}^n E_{lj} W_l \left( \frac{1}{N_j^{(i)}} - \frac{1}{n} \sum_{\iota \in [n], \iota \neq l} \frac{1}{N_j^{(\iota)}} \right) - \frac{E_{ij}}{N_j^{(i)}} W_i. \end{aligned} \quad (\text{SA-33})$$

By a decomposition argument,

$$\begin{aligned}
\frac{1}{N_j^{(i)}} - \frac{1}{n} \sum_{\iota \in [n], \iota \neq i} \frac{1}{N_j^{(\iota)}} &= \frac{1}{N_j^{(i)}} - \frac{1}{n-1} \sum_{\iota \in [n], \iota \neq i} \frac{1}{N_j^{(\iota)}} + \frac{1}{(n-1)n} \sum_{\iota \in [n], \iota \neq i} \frac{1}{N_j^{(\iota)}} \\
&= \frac{1}{n-1} \sum_{\iota \in [n], \iota \neq i} \frac{E_{ji} - E_{j\iota}}{N_j^{(i)} N_j^{(\iota)}} + \frac{1}{(n-1)n} \sum_{\iota \in [n], \iota \neq i} \frac{1}{N_j^{(\iota)}} \\
&= n^{-1} (n\rho_n)^{-1} \frac{E_{ij} - \rho_n g(U_j)}{\rho_n g(U_j)^2} + \frac{1}{(n-1)n} \sum_{\iota \in [n], \iota \neq i} \frac{1}{N_j^{(\iota)}}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{l=1}^n E_{lj} W_l \left( \frac{1}{N_j^{(i)}} - \frac{1}{n} \sum_{\iota \in [n], \iota \neq i} \frac{1}{N_j^{(\iota)}} \right) \\
&= (n\rho_n)^{-1} \frac{E_{ij} - \rho_n g(U_j)}{\rho_n g(U_j)^2} \frac{1}{n} \sum_{l=1}^n E_{lj} W_l + \frac{\sum_{l=1}^n E_{lj} W_l}{N_j^{(i)}} O_{\psi_{2,tc}}((n\rho_n)^{-\frac{3}{2}}) \\
&\quad + \frac{1}{n-1} \sum_{\iota \in [n], \iota \neq i} \frac{\sum_{l=1}^n E_{lj} W_l}{n N_j^{(\iota)}}.
\end{aligned}$$

Condition on  $U_j$ ,  $(E_{lj} W_l : l \neq j)$  are i.i.d mean-zero, hence Bernstein inequality gives  $\frac{1}{n} \sum_{l=1}^n E_{lj} W_l = O_{\psi_2}(\sqrt{n^{-1}\rho_n}) + O_{\psi_1}(n^{-1})$ , which implies

$$\begin{aligned}
(n\rho_n)^{-1} \frac{E_{ij} - \rho_n g(U_j)}{\rho_n g(U_j)^2} \frac{1}{n} \sum_{l=1}^n E_{lj} W_l &= O_{\psi_2}((n\rho_n)^{-\frac{3}{2}}) + O_{\psi_1}((n\rho_n)^{-2}), \\
\frac{1}{n-1} \sum_{\iota \in [n], \iota \neq i} \frac{\sum_{l=1}^n E_{lj} W_l}{n N_j^{(\iota)}} &= O_{\psi_2}(n^{-\frac{3}{2}} \rho_n^{-\frac{1}{2}}) + O_{\psi_1}(n^{-2}).
\end{aligned}$$

Putting back into Equation (SA-33),

$$\frac{M_j}{N_j^{(i)}} - \frac{1}{n} \sum_{\iota \in [n]} \frac{M_j}{N_j^{(\iota)}} = -\frac{E_{ij}}{N_j^{(i)}} W_i + O_{\psi_1}((n\rho_n)^{-\frac{3}{2}}).$$

Looking at contribution from the first order term in Taylor expanding  $h_j(1, \cdot)$  to  $\tau_{(i)}^b - \bar{\tau}^b$  in Equation (SA-30),

$$\begin{aligned}
&\frac{1}{n} \sum_{j \in [n]} \partial_2 h(1, 0) \frac{T_j}{1/2} \left[ \frac{M_j}{N_j^{(i)}} - \frac{1}{n} \sum_{\iota \in [n]} \frac{M_j}{N_j^{(\iota)}} \right] \\
&= - \sum_{j \in [n]} \partial_2 h(1, 0) W_i \frac{1}{n} \frac{E_{ij}}{N_j^{(i)}} \frac{T_j}{1/2} + O_{\psi_{1,tc}}((n\rho_n)^{-\frac{3}{2}}) \\
&= - W_i \frac{1}{n} \sum_{j \in [n]} \partial_2 h(1, 0) \frac{E_{ij}}{n\rho_n g(U_j)} \frac{T_j}{1/2} - W_i \frac{1}{n} \sum_{j \in [n]} \partial_2 h(1, 0) \frac{E_{ij}}{N_j^{(i)}} \frac{n\rho_n g(U_j) - N_j}{n\rho_n g(U_j)} \frac{T_j}{1/2} \\
&\quad + O_{\psi_{1,tc}}((n\rho_n)^{-\frac{3}{2}}) \\
&= - W_i \frac{1}{n} \sum_{j \in [n]} \partial_2 h(1, 0) \frac{E_{ij}}{n\rho_n g(U_j)} \frac{T_j}{1/2} + O_{\psi_{1,tc}}((n\rho_n)^{-\frac{3}{2}}).
\end{aligned}$$

Since  $(E_{ij}T_j/g(U_j) : j \in [n])$  are independent condition on  $U_i$ , standard concentration inequality gives

$$\begin{aligned}
& \frac{1}{n} \sum_{j \in [n]} \partial_2 h(1, 0) \frac{T_j}{1/2} \left[ \frac{M_j}{N_j^{(i)}} - \frac{1}{n} \sum_{\iota \in [n]} \frac{M_j}{N_j^{(\iota)}} \right] \\
&= -W_i \frac{1}{n} \sum_{j \in [n]} \partial_2 h(1, 0) \frac{E_{ij}}{n \rho_n g(U_j)} \frac{T_j}{1/2} + O_{\psi_{1,tc}}((n\rho_n)^{-\frac{3}{2}}) \\
&= -W_i \partial_2 h(1, 0) \frac{1}{n} \sum_{j \in [n]} \frac{E_{ij}}{n \rho_n g(U_j)} \frac{T_j}{1/2} + O_{\psi_{1,tc}}((n\rho_n)^{-\frac{3}{2}}) \\
&= -\partial_2 h(1, 0) \frac{W_i}{n} \mathbb{E} \left[ \frac{E_{ij}}{\rho_n g(U_j)} \middle| U_i \right] + O_{\psi_{1,tc}}((n\rho_n)^{-\frac{3}{2}}).
\end{aligned}$$

Since we assumed  $\partial_2 h(1, 0) = \partial_2 f(1, 0) + o_{\mathbb{P}}(1) = \partial_2 f_j(1, 0) + o_{\mathbb{P}}(1)$  where

$$\begin{aligned}
& \frac{1}{n} \sum_{j \in [n]} \partial_2 h(1, 0) \frac{T_j}{1/2} \left[ \frac{M_j}{N_j^{(i)}} - \frac{1}{n} \sum_{\iota \in [n]} \frac{M_j}{N_j^{(\iota)}} \right] \\
&= -\frac{W_i}{n} \mathbb{E} \left[ \frac{E_{ij} \partial_2 f_j(1, 0)}{\rho_n g(U_j)} \middle| U_i \right] + O_{\psi_{1,tc}}((n\rho_n)^{-\frac{3}{2}}) + o_{\mathbb{P}}(n^{-1}).
\end{aligned}$$

Together with the leading term in Equation (SA-30), we have

$$\begin{aligned}
& n \sum_{i \in [n]} \left( \frac{1}{n} \sum_{j \in [n]} \partial_2 h_j(1, 0) \frac{T_j}{1/2} \left[ \frac{M_j}{N_j^{(i)}} - \frac{1}{n} \sum_{\iota \in [n]} \frac{M_j}{N_j^{(\iota)}} \right] + \tau_{(i)}^a - \bar{\tau}^a \right) \\
& \quad \left( \frac{2}{n_q} \sum_{j \in \mathcal{I}_q} \partial_2 h_j(1, 0) \frac{T_j}{\theta_q} \left[ \frac{M_j}{N_j^{(i)}} - \frac{1}{n} \sum_{\iota \in [n]} \frac{M_j}{N_j^{(\iota)}} \right] + \tau_{(i)}^a - \bar{\tau}^a \right) \\
&= \frac{n}{n^2} \sum_{i \in [n]} \left( \mathbb{E} \left[ \frac{E_{ij} \partial_2 f_j(1, 0)}{\rho_n g(U_j)} \middle| U_i \right] + \frac{f_i(1, 0)}{1/2} (T_i - 1/2) \right) \\
& \quad \left( \mathbb{E} \left[ \frac{E_{ij} \partial_2 f_j(1, 0)}{\rho_n g(U_j)} \middle| U_i \right] + \frac{f_i(1, 0)}{1/2} (T_i - 1/2) \right) + O_{\psi_{1,tc}}((n\rho_n^3)^{-1}) + o_{\mathbb{P}}(1) \\
&= \frac{n_l^2}{n^2} \mathbb{E} \left[ \left( \mathbb{E} \left[ \frac{E_{ij} \partial_2 f_j(1, 0)}{\rho_n g(U_j)} \middle| U_i \right] + \frac{f_i(1, 0)}{1/2} (T_i - 1/2) \right) \right. \\
& \quad \left. \left( \mathbb{E} \left[ \frac{E_{ij} \partial_2 f_j(1, 0)}{\rho_n g(U_j)} \middle| U_i \right] + \frac{f_i(1, 0)}{1/2} (T_i - 1/2) \right) \right] + O_{\psi_{1,tc}}((n\rho_n^3)^{-1}) + o_{\mathbb{P}}(1) \\
&= \mathbf{e}_s^\top \mathbb{E}[\mathbf{S}_\ell \mathbf{S}_\ell^\top] \mathbf{e}_q + O_{\psi_{1,tc}}((n\rho_n^3)^{-1}) + o_{\mathbb{P}}(1).
\end{aligned}$$

**Part 2: Higher Order Terms** For the second order terms, first notice that if  $l \notin [n]$ , then

$$\begin{aligned}
& \left( \frac{M_j}{N_j^{(i)}} \right)^2 - \frac{1}{n} \sum_{\iota \in [n], \iota \neq l} \left( \frac{M_j}{N_j^{(\iota)}} \right)^2 \\
&= \frac{1}{n} \sum_{\iota \in [n], \iota \neq l} \left( \frac{M_j}{N_j^{(i)}} + \frac{M_j}{N_j^{(\iota)}} \right) \frac{M_j (E_{ij} - E_{\iota j}) - (E_{ij} W_i - E_{\iota j} W_\iota) N_j + E_{ij} E_{\iota j} (W_i - W_\iota)}{N_j^{(i)} N_j^{(\iota)}} \\
&= O_{\psi_{2,tc}}((n\rho_n)^{-\frac{3}{2}}),
\end{aligned}$$

where we have used  $(M_j/N_j)_\iota = O_{\psi_2}((n\rho_n)^{-\frac{1}{2}})$  and  $N_j^{-1} = O_{\psi_2}((n\rho_n)^{-1})$ . If  $l \in [n]$ , then again

$$\begin{aligned} & \left( \frac{M_j}{N_j(i)} \right)^2 - \frac{1}{n} \sum_{\iota \in [n], \iota \neq l} \left( \frac{M_j}{N_j(\iota)} \right)^2 \\ &= \left( \frac{M_j}{N_j(i)} \right)^2 - \frac{1}{n-1} \sum_{\iota \in [n], \iota \neq l} \left( \frac{M_j}{N_j(\iota)} \right)^2 + \frac{1}{(n-1)n} \sum_{\iota \in [n], \iota \neq l} \left( \frac{M_j}{N_j(\iota)} \right)^2 \\ &= O_{\psi_{2,tc}}((n\rho_n)^{-\frac{3}{2}}). \end{aligned}$$

Hence

$$\begin{aligned} & n \sum_{i \in [n]} \left( \partial_{2,2}h(1,0) \frac{2}{n} \sum_{j \in [n]} T_j \left[ \left( \frac{M_j}{N_j(i)} \right)^2 - \frac{1}{n} \sum_{\iota \in [n]} \left( \frac{M_j}{N_j(\iota)} \right)^2 \right] \right) \\ & \quad \left( \partial_{2,2}h(1,0) \frac{2}{n_q} \sum_{j \in \mathcal{I}_q} T_j \left[ \left( \frac{M_j}{N_j(i)} \right)^2 - \frac{1}{n} \sum_{\iota \in [n]} \left( \frac{M_j}{N_j(\iota)} \right)^2 \right] \right) = O_{\psi_{2,tc}}((n\rho_n^3)^{-1}). \end{aligned}$$

For the third order residual, observe that  $(\frac{M_j}{N_j(\iota)})^3 = O_{\psi_2}((n\rho_n)^{-3/2})$ . Then

$$\begin{aligned} & n \sum_{i \in [n]} \left( \frac{2}{n} \sum_{j \in [n]} T_j \left[ \partial_{2,2,2}h(1, \xi_{j,i}^*) \left( \frac{M_j}{N_j(i)} \right)^3 - \frac{1}{n} \sum_{\iota \in [n]} \partial_{2,2,2}h(1, \xi_{j,\iota}^*) \left( \frac{M_j}{N_j(\iota)} \right)^3 \right] \right) \\ & \quad \left( \frac{2}{n_q} \sum_{j \in \mathcal{I}_q} T_j \left[ \partial_{2,2,2}h(1, \xi_{j,i}^*) \left( \frac{M_j}{N_j(i)} \right)^3 - \frac{1}{n} \sum_{\iota \in [n]} \partial_{2,2,2}h(1, \xi_{j,\iota}^*) \left( \frac{M_j}{N_j(\iota)} \right)^3 \right] \right) \\ & = O_{\psi_{2,tc}}((n\rho_n^3)^{-1}). \end{aligned}$$

The conclusion then follows from Equations (SA-26), (SA-30) and (SA-31).

### SA-7.5 Proof of Lemma SA-2

Define  $\mathbf{r}(x) = (1, x)^\top$ . Denote  $\pi = \mathbb{E}[W_i] = 2\mathbb{E}[T_i] - 1$ . Then

**Case 1:**  $\beta < 1$

First, consider the gram-matrix. Take  $\zeta_i := \sqrt{n\rho_n}(\frac{M_i}{N_i} - \pi)$ . Then  $1 \lesssim \mathbb{V}[\zeta_i] \lesssim 1$ . Take  $b_n = \sqrt{n\rho_n}h_n$ . Take

$$\mathbf{B}_n := \frac{1}{nb_n} \sum_{i=1}^n \mathbf{r}\left(\frac{\zeta_i}{b_n}\right) \mathbf{r}\left(\frac{\zeta_i}{b_n}\right)^\top K\left(\frac{\zeta_i}{b_n}\right),$$

where  $\mathbf{r} : \mathbb{R} \rightarrow \mathbb{R}^2$  is given by  $\mathbf{r}(u) = (1, u)^\top$ . Take  $Q$  to be the probability measure of  $\zeta_i$  given  $\mathbf{E}$ . Then

$$\mathbf{B} := \mathbb{E}[\mathbf{B}_n | \mathbf{E}] = \begin{bmatrix} \int_{-\infty}^{\infty} \frac{1}{b_n} K\left(\frac{x}{b_n}\right) dQ(x) & \int_{-\infty}^{\infty} \frac{x}{b_n} \frac{1}{b_n} K\left(\frac{x}{b_n}\right) dQ(x) \\ \int_{-\infty}^{\infty} \frac{x}{b_n} \frac{1}{b_n} K\left(\frac{x}{b_n}\right) dQ(x) & \int_{-\infty}^{\infty} \left(\frac{x}{b_n}\right)^2 \frac{1}{b_n} K\left(\frac{x}{b_n}\right) dQ(x) \end{bmatrix}.$$

In particular,  $\lambda_{\min}(\mathbf{B}) \gtrsim 1$ . Now we want to show each entry of  $\mathbf{B}_n$  converge to those of  $\mathbf{B}$ . Take

$$F_{p,q}(\mathbf{W}) := \mathbf{e}_p^\top \mathbf{B}_n \mathbf{e}_q = \frac{1}{nb_n} \sum_{i=1}^n \left( \frac{\zeta_i}{b_n} \right)^{p+q} K\left(\frac{\zeta_i}{b_n}\right), \quad p, q \in \{0, 1\}.$$

Denote  $\partial_j$  to be the partial derivative w.r.p to  $W_j$ . Since  $K$  is Lipschitz with bounded support,

$$|\partial_j F_{p,q}(\mathbf{W})| \lesssim \frac{1}{b_n^2} \frac{1}{n} \sum_{i=1}^n \left| \partial_j \left( \frac{M_i}{N_i} - \pi \right) \right| \lesssim \frac{1}{b_n^2} \frac{1}{n} \sum_{i=1}^n \frac{E_{ij}}{N_i}. \quad (\text{SA-34})$$

Condition on  $\mathbf{E}$ ,

$$F_{p,q}(\mathbf{W}) = \mathbb{E}[F_{p,q}(\mathbf{W})|\mathbf{E}] + O_{\psi_2} \left( \sum_{j=1}^n |\partial_j F_{p,q}(\mathbf{W})|^2 \right) = \mathbf{e}_p^\top \mathbf{B} \mathbf{e}_q + O_{\psi_2} \left( \frac{1}{nb_n^4} \frac{1}{n} \sum_{j=1}^n \left( \sum_{i=1}^n \frac{E_{ij}}{N_i} \right)^2 \right).$$

Hence for all  $p, q \in \{0, 1\}$ ,

$$\mathbf{e}_p^\top \mathbf{B}_n \mathbf{e}_q = \mathbf{e}_p^\top \mathbf{B} \mathbf{e}_q + O_{\psi_2}((nb_n^4)^{-1}).$$

Since both  $\mathbf{B}_n$  and  $\mathbf{B}$  are two by two matrices,  $\|\mathbf{B}_n - \mathbf{B}\|_{\text{op}} \lesssim O_{\psi_2}((nb_n^4)^{-1})$ . By Weyl's Theorem,

$$|\lambda_{\min}(\mathbf{B}_n) - \lambda_{\min}(\mathbf{B})| \leq \|\mathbf{B}_n - \mathbf{B}\|_{\text{op}} \lesssim (nb_n^4)^{-1}, \quad (\text{SA-35})$$

and together with  $\lambda_{\min}(\mathbf{B}) \gtrsim 1$ , implies  $\lambda_{\min}(\mathbf{B}_n) \gtrsim 1$ . Take

$$\boldsymbol{\Sigma}_n := \frac{1}{nb_n^2} \sum_{i=1}^n \mathbf{r} \left( \frac{\zeta_i}{b_n} \right) \mathbf{r} \left( \frac{\zeta_i}{b_n} \right)^\top K^2 \left( \frac{\zeta_i}{b_n} \right) \mathbb{V}[Y_i | \zeta_i].$$

Hence variance can be bounded by

$$\mathbb{V}[\widehat{\gamma}_0 | \mathbf{E}, \mathbf{W}] = \mathbf{e}_0^\top \mathbf{B}_n^{-1} \boldsymbol{\Sigma}_n \mathbf{B}_n^{-1} \mathbf{e}_0 \lesssim (nb_n)^{-1}, \quad (\text{SA-36})$$

$$\mathbb{V}[\widehat{\gamma}_1 | \mathbf{E}, \mathbf{W}] = n\rho_n \mathbf{e}_1^\top \mathbf{B}_n^{-1} \boldsymbol{\Sigma}_n \mathbf{B}_n^{-1} \mathbf{e}_1 \lesssim (n\rho_n)(nb_n^3)^{-1} = \rho_n b_n^{-3}. \quad (\text{SA-37})$$

Next, consider the bias term. Since  $f(1, \cdot) \in C^2$ , whenever  $|\frac{M_i}{N_i} - \pi| \leq h_n = (n\rho_n)^{-1/2} b_n$ ,

$$\begin{aligned} f(1, M_i/N_i) &= f(1, \pi) + \partial_2 f(1, \pi) \left( \frac{M_i}{N_i} - \pi \right) + O \left( \left( \frac{M_i}{N_i} - \pi \right)^2 \right) \\ &= f(1, \pi) + \partial_2 f(1, \pi) \left( \frac{M_i}{N_i} - \pi \right) + O((n\rho_n)^{-1} b_n^2). \end{aligned}$$

Hence using the fourth and third lines above respectively,

$$\begin{aligned} \mathbb{E}[\widehat{\gamma}_0 | \mathbf{E}, \mathbf{W}] &= \mathbf{e}_0^\top \mathbf{B}_n^{-1} \left[ \frac{1}{nb_n} \sum_{i=1}^n \mathbf{r} \left( \frac{\zeta_i}{b_n} \right) K \left( \frac{\zeta_i}{b_n} \right) f \left( 1, \frac{M_i}{N_i} \right) \right] \\ &= \mathbf{e}_0^\top \mathbf{B}_n^{-1} \left[ \frac{1}{nb_n} \sum_{i=1}^n \mathbf{r} \left( \frac{\zeta_i}{b_n} \right) K \left( \frac{\zeta_i}{b_n} \right) \left( \mathbf{r} \left( \frac{\zeta_i}{b_n} \right)^\top (f(1, \pi), \frac{1}{\sqrt{n\rho_n}} \partial_2 f(1, \pi))^\top + O_{\psi_2}((n\rho_n)^{-\frac{1}{2}}) \right) \right] \\ &= f(1, \pi) + O_{\psi_2}((n\rho_n)^{-\frac{1}{2}}), \\ \mathbb{E}[\widehat{\gamma}_1 | \mathbf{E}, \mathbf{W}] &= \sqrt{n\rho_n} \mathbf{e}_1^\top \mathbf{B}_n^{-1} \left[ \frac{1}{nb_n} \sum_{i=1}^n \mathbf{r} \left( \frac{\zeta_i}{b_n} \right) K \left( \frac{\zeta_i}{b_n} \right) f \left( 1, \frac{M_i}{N_i} \right) \right] \\ &= \sqrt{n\rho_n} \mathbf{e}_1^\top \mathbf{B}_n^{-1} \left[ \frac{1}{nb_n} \sum_{i=1}^n \mathbf{r} \left( \frac{\zeta_i}{b_n} \right) K \left( \frac{\zeta_i}{b_n} \right) \left( \mathbf{r} \left( \frac{\zeta_i}{b_n} \right)^\top (f(1, \pi), \frac{1}{\sqrt{n\rho_n}} \partial_2 f(1, \pi))^\top + O_{\psi_2}((n\rho_n)^{-1}) \right) \right] \\ &= \partial_2 f(1, \pi) + O_{\psi_2}((n\rho_n)^{-\frac{1}{2}}), \end{aligned} \quad (\text{SA-38})$$

Putting together Equations (SA-36) and (SA-38),

$$\widehat{\gamma}_0 - \gamma_0 = O_{\mathbb{P}}((n\rho_n)^{-\frac{1}{2}} + (nb_n)^{-\frac{1}{2}}), \quad \widehat{\gamma}_1 - \gamma_1 = O_{\mathbb{P}}((n\rho_n)^{-\frac{1}{2}} + \rho_n b_n^{-3}).$$

Hence any  $b_n$  such that  $b_n = \Omega(n^{-1/4} + \rho_n^{1/3})$  will make  $(\widehat{\gamma}_0, \widehat{\gamma}_1)$  a consistent estimator for  $(\gamma_0, \gamma_1)$ . For any  $0 \leq \rho_n \leq 1$  such that  $n\rho_n \rightarrow \infty$ , such a sequence  $b_n$  exists.

**Case 2:**  $\beta = 1$

The order  $\frac{M_i}{N_i}$  is  $n^{-1/4}$  if  $\liminf_{n \rightarrow \infty} n\rho_n^2 > c$  for some  $c > 0$ ; and is  $(n\rho_n)^{-1/2}$  if  $n\rho_n^2 = o(1)$ . We consider these two cases separately.

**Case 2.1:**  $\liminf_{n \rightarrow \infty} n\rho_n^2 > c$  for some  $c > 0$  Take  $\eta_i = n^{\frac{1}{4}}(\frac{M_i}{N_i} - \pi)$ . Take  $d_n = n^{1/4}h_n$ . And with the same  $\mathbf{r}$  defined in Case 1,

$$\mathbf{D}_n := \frac{1}{nd_n} \sum_{i=1}^n \mathbf{r}\left(\frac{\eta_i}{d_n}\right) \mathbf{r}\left(\frac{\eta_i}{d_n}\right)^{\top} K\left(\frac{\eta_i}{d_n}\right), \quad \mathbf{D} = \mathbb{E}[\mathbf{D}_n].$$

Under the assumption  $\liminf_{n \rightarrow \infty} n\rho_n^2 \leq c$  for some  $c > 0$ , we have  $1 \lesssim \mathbb{V}[\eta_i] \lesssim 1$ . Hence  $\lambda_{\min}(\mathbf{D}) \gtrsim 1$ . To study the convergence between  $\mathbf{D}_n$  and  $\mathbf{D}$ , again consider for  $p, q \in \{0, 1\}$ ,

$$G_{p,q}(\mathbf{W}) := \mathbf{e}_p^{\top} \mathbf{D}_n \mathbf{e}_q = \frac{1}{nd_n} \sum_{i=1}^n \left(\frac{\eta_i}{d_n}\right)^{p+q} K\left(\frac{\eta_i}{d_n}\right) = \frac{1}{n^{5/4}h_n} \sum_{i=1}^n \left(h_n^{-1}\left(\frac{M_i}{N_i} - \pi\right)\right)^{p+q} K\left(h_n^{-1}\left(\frac{M_i}{N_i} - \pi\right)\right).$$

Still let  $\mathbf{U}_n$  be the latent variable from Lemma SA-1,  $W_i$ 's are independent conditional on  $\mathbf{U}_n$ . Hence by similar argument as Equation (SA-34), we can show

$$G_{p,q}(\mathbf{W}) = \mathbb{E}[G_{p,q}(\mathbf{W})|\mathbf{U}_n, \mathbf{E}] + O_{\psi_2}((nd_n^4)^{-1}).$$

Moreover, recall we denote by  $\omega_i \in [k]$  the block unit  $i$  belongs to, then

$$\mathbb{E}[G_{p,q}(\mathbf{W})|\mathbf{U}_n, \mathbf{E}] = \sum_{\mathbf{W} \in \{-1, 1\}^n} \prod_{i=1}^n p(U_{\omega_i})^{W_i} (1 - p(U_{\omega_i}))^{1-W_i} G_{p,q}(\mathbf{W}),$$

$p(U_i) = \mathbb{P}(W_i = 1|U_i) = \frac{1}{2}(\tanh(\sqrt{\beta_{\ell}/n}U_n + h_{\ell}) + 1)$ ,  $i \in \mathcal{I}_{\ell}$ . Take the derivative term by term,

$$\partial_{U_{\ell}} \mathbb{E}[G_{p,q}(\mathbf{W})|\mathbf{U}_n, \mathbf{E}] = \sum_{j \in \mathcal{I}_{\ell}} \mathbb{E}_{\mathbf{W}_{-j}} [G_{p,q}(W_j = 1, W_{-j}) - G_{p,q}(W_j = -1, W_{-j})] p'(U_{\ell}).$$

Using Lipschitz property of  $x \mapsto (x/h_n)^{p+q} K(x/h_n)$ ,

$$|G_{p,q}(W_j = 1, W_{-j}) - G_{p,q}(W_j = -1, W_{-j})| \lesssim \frac{1}{n^{5/4}h_n} \sum_{i=1}^n \frac{1}{h_n} \frac{E_{ij}}{N_i}.$$

Hence for all  $\ell \in \mathcal{C}$ ,

$$|\partial_{U_{\ell}} \mathbb{E}[G_{p,q}(\mathbf{W})|\mathbf{U}_n, \mathbf{E}]| \lesssim \sum_{j \in \mathcal{I}_{\ell}} \frac{1}{n^{5/4}h_n} \sum_{i=1}^n \frac{1}{h_n} \frac{E_{ij}}{N_i} \|p'\|_{\infty} \lesssim \frac{1}{n^{3/4}h_n^2}.$$

Moreover, for all  $\ell \in \mathcal{C}$ ,  $\|U_\ell\|_{\varphi_2} \lesssim n^{1/4}$ . Together, this gives

$$\mathbb{E}[G_{p,q}(\mathbf{W})|\mathbf{U}_n, \mathbf{E}] - \mathbb{E}[G_{p,q}(\mathbf{W})|\mathbf{E}] = O_{\mathbb{P}}((n^{1/2}h_n^2)^{-1}) = O_{\mathbb{P}}(d_n^{-2}).$$

Hence if we take  $d_n \gg 1$  (which implies  $nd_n^4 \gg 1$ ), then  $G_{p,q}(\mathbf{W}) = \mathbb{E}[G_{p,q}(\mathbf{W})|\mathbf{E}] + o_{\mathbb{P}}(1)$ , implying  $\|\mathbf{D}_n - \mathbf{D}\|_2 = o_{\mathbb{P}}(1)$  and  $\lambda_{\min}(\mathbf{D}_n) - \lambda_{\min}(\mathbf{D}) = o_{\mathbb{P}}(1)$ , making  $\lambda_{\min}(\mathbf{D}_n) \gtrsim_{\mathbb{P}} 1$ . Take

$$\boldsymbol{\Upsilon}_n := \frac{1}{nd_n^2} \sum_{i=1}^n \mathbf{r}\left(\frac{\eta_i}{d_n}\right) \mathbf{r}\left(\frac{\eta_i}{d_n}\right)^\top K^2\left(\frac{\eta_i}{d_n}\right) \mathbb{V}[Y_i|\eta_i].$$

Hence variance can be bounded by

$$\mathbb{V}[\widehat{\gamma}_0|\mathbf{E}, \mathbf{W}] = \mathbf{e}_0^\top \mathbf{D}_n^{-1} \boldsymbol{\Upsilon}_n \mathbf{D}_n^{-1} \mathbf{e}_0 \lesssim (nd_n)^{-1}, \quad (\text{SA-39})$$

$$\mathbb{V}[\widehat{\gamma}_1|\mathbf{E}, \mathbf{W}] = n^{1/2} \mathbf{e}_1^\top \mathbf{D}_n^{-1} \boldsymbol{\Upsilon}_n \mathbf{D}_n^{-1} \mathbf{e}_1 \lesssim n^{1/2} (nd_n^3)^{-1} = n^{-1/2} d_n^{-3}. \quad (\text{SA-40})$$

By similar argument as in Case 1, assume  $d_n \gg 1$ , we can show

$$\mathbb{E}[\widehat{\gamma}_0|\mathbf{E}] - \gamma_0 = O(n^{-1/4} + n^{-1/2}d_n^2), \quad \mathbb{E}[\widehat{\gamma}_1|\mathbf{E}] - \gamma_1 = O(n^{-1/4}d_n^2).$$

Hence if we choose  $d_n$  such that  $1 \ll d_n \ll n^{1/8}$ , then  $(\widehat{\gamma}_0, \widehat{\gamma}_1)$  is a consistent estimator for  $(\gamma_0, \gamma_1)$ . The only assumption we made for the existence of such a  $d_n$  is  $\liminf_{n \rightarrow \infty} n\rho_n^2 \geq c$  for some  $c > 0$ .

**Case 2.2:**  $n\rho_n^2 = o(1)$  Take  $\eta_i := \sqrt{n\rho_n}(\frac{M_i}{N_i} - \pi)$ ,  $d_n = \sqrt{n\rho_n}h_n$ . By similar decomposition based on latent variables, we can show if  $n\rho_n \rightarrow \infty$  as  $n \rightarrow \infty$ , then there exists  $h_n$  such that  $(\widehat{\gamma}_0, \widehat{\gamma}_1)$  is a consistent estimator for  $(\gamma_0, \gamma_1)$ .

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